

Designing Fractal Curves with Five-Fold Rotational Symmetry Using the Complex Number Golden Ratio

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Abstract

The regular pentagon cannot tile the plane, although many polygons related to the pentagon can tile the plane, including those that admit Penrose tilings. Plane-filling fractal curves are closely-related to tilings, and so the question is asked: are there pentagon-based fractal curves that can fill the plane? This paper introduces a method for designing fractal curves with local five-fold rotational symmetry, using a special mathematical tool: the first 10th root of unity—a complex number that easily generates the golden ratio. It can be raised to integer powers for rotation, and can be scaled by its own derived golden ratio, which itself can be raised to integer powers to determine the fractal scaling factor. Building upon previous work in categorizing fractal curves with complex integers, this paper incorporates inspiration from some new developments in tiling and fractals.

Introduction

Those of us who are helplessly in love with fractal geometry are easily hypnotized by the pentagram—constructed by connecting the vertices of a regular pentagon with line segments. Each internal line segment is crossed twice by two other segments (Figure 1(a)). The sub-segments that result from these crossings conveniently demonstrate the golden ratio ϕ (1.618033...). The self-similar nature of ϕ makes it quite friendly with fractal geometry. In this paper, I will describe a method for constructing fractal curves using a special class of complex numbers that express five-fold rotational symmetry. This symmetry can be seen locally on many scales, but not necessarily in the overall shape (as in Figure 1(b)). A few approximations of plane-filling curves (iterated to a limited number of recursive levels) are shown in Figures 1(c) and 1(d). In this paper, the term “plane-filling” refers to these approximations.

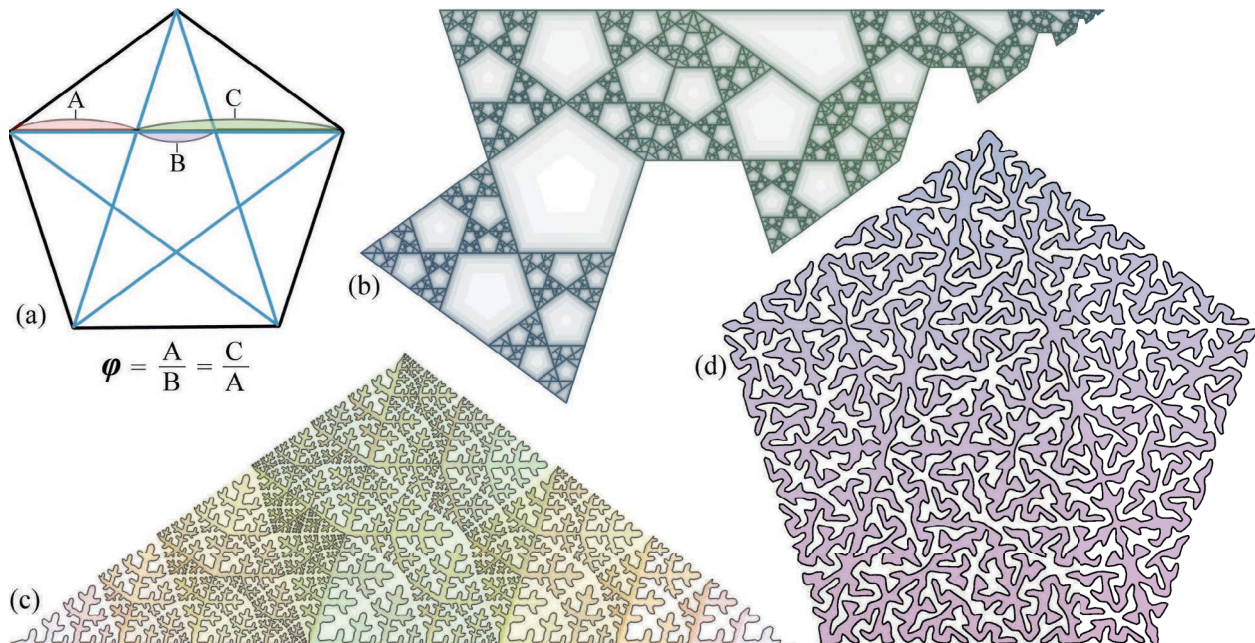


Figure 1: (a) the golden ratio in the pentagram; (b) a color-filled gasket curve; (c) a stylized plane-filling curve; (d) a plane-filling curve that fills a pentagon.

This method extends a previous method I was using for categorizing and constructing plane-filling fractal curves [18][19], in which all curves conform to one of two kinds of integer domains in the complex plane (the Gaussian integers, corresponding to the square lattice, and the Eisenstein integers, corresponding to the triangular lattice). This extended method is the result of a collaboration with Pautze [10]. It includes a new kind of integer domain that does not conform to a regular tiling. This domain has some curious inhabitants—including the celebrated Penrose tilings. (This method can also generate novel Penrose tilings, assuming the curve does not cross itself and is plane-filling; having a fractal dimension of 2).

The key lies in a unique complex number: the *first 10th root of unity*. It can be expressed in polar form as a vector with a length of 1 and an angle of 36 degrees. This number is shown in the middle of Figure 2 as ζ . Complex numbers of magnitude 1 (on the unit circle) that do not lie on the real axis rotate counterclockwise when raised by an exponent—and they remain on the unit circle. In the case of ζ , raising it to an integer power n changes the angle of the vector to $36n$ degrees. Raising it to the power of 0 sets the angle to 0. All integer powers of ζ lie on vectors that cut the unit pie into ten equal slices.

The first 10th root of unity is one of 10 belonging to the *10th cyclotomic field*. A special property of this number is that you can get ϕ simply by summing it with its conjugate (its reflection about the real number line—which is the same as ζ^9). This is shown at the bottom-right of Figure 2, where these summed numbers appear as vectors forming a triangular bump, with the right-side of the bump lying on the real number line, at a distance of ϕ from the origin. In this context, the golden ratio can be understood as the sum of two *algebraic integers*. Let’s call this the “complex number golden ratio”, because it is born in the complex plane, and it is useful to consider it in this context.

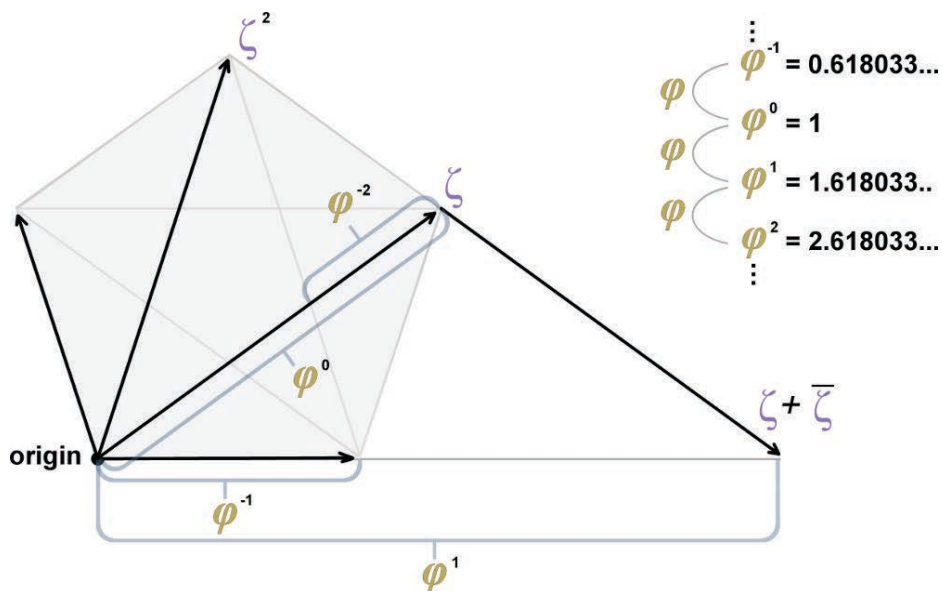


Figure 2: Powers of ζ and powers of ϕ in the complex plane.

Golden Powers

Here’s an interesting property of the golden ratio: it can be raised by integer powers to create golden ratio *ratios*. Consider the series at the top-right of Figure 2: any two consecutive powers of ϕ have a ratio of ϕ : ($\phi^{p+2} = \phi^{p+1} + \phi^p$). Thus, any integer power of ϕ can be expressed as sums of other integer powers of ϕ . A few powers of ϕ are shown within the diagram, referring to various segment lengths in the pentagram. And so: we can use ζ^n to represent the angle of a line segment, and we can use ϕ^p to represent its length. Every segment in a fractal curve generated with this method can be expressed in the form: $\zeta^n \phi^p$, where n and p are integers. These integers can be zero, positive, or negative. They serve as the atomic alphabet for encoding a very large class of fractal curves. I use this alphabet—and a corresponding set of rules for recursive replacement—as a tool for algorithmic visual design.

Background

Five-fold-symmetrical designs appear throughout art history, perhaps most famously in the ornamentation of Medieval Islamic architecture (Figure 3(a)). Kepler [7] and Dürer [2] explored ways to tile the plane with pentagonal geometry, undoubtedly experiencing some of the frustration mixed with determination that we experience today when attempting to tile the plane with pentagons. (This “trouble with five” is nicely explained by Craig Kaplan [6]). The subject of pentagonal tiling gained more attention in the previous century with the discovery of Penrose tilings [13][14]. These tilings are aperiodic: there is no way to arrange the tiles into a pattern that repeats itself on an arbitrarily large scale (no translation maps the tiling to itself). From the perspective of visual design, aperiodic tilings provide a wonderful mathematical/artistic canvas, having a complex mixture of symmetry and asymmetry, at multiple scales.

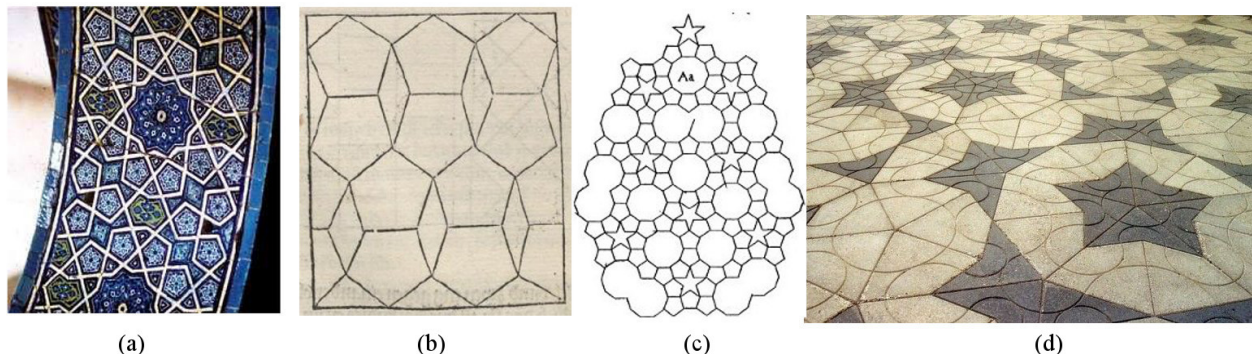


Figure 3: *Examples of tilings based on pentagons: (a) Darb-e Imam Shrine, Iran, (b) Dürer tiling, (c) Kepler tiling, (d) Penrose tiling.*

Most of the classic fractal curves, such as those introduced by Peano [12], Hilbert [5], Gosper [20], and Mandelbrot [8] can be fit onto regular square or triangular lattices. A few notable non-crossing fractal curves that conform to five-fold rotational symmetry include McWorter’s Pentigree [3] and a curve identified by Henle [4] that fills the pentagon (Figure 4a and b). T. D. Taylor created “golden Koch curves” using fractal trees [17].

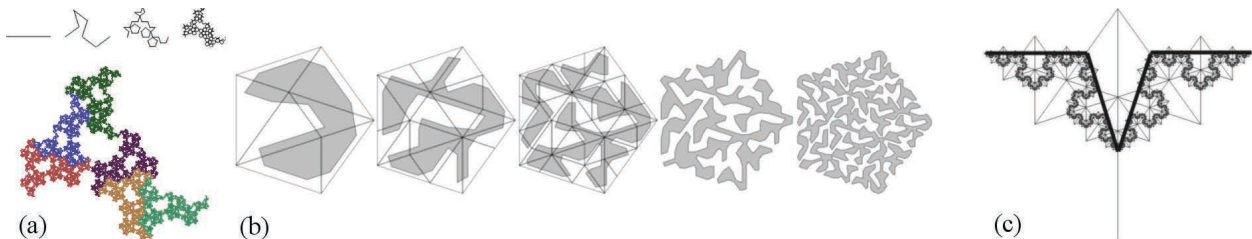


Figure 4: *(a) McWorter’s Pentigree; (b) Henle’s pentagon-filling curve; (c) Taylor’s golden Koch curve.*

Development

A general approach to finding plane-filling curves in regular grids using L-systems is described by Arndt [1]. Recall that the only regular polygons that can tile the plane are triangles, squares and hexagons. The vertices of these tiles form lattice-points. Since the hexagonal lattice can be seen as a subset of the triangular lattice, it is not considered. In the process of working out the previous method, I learned about the Gaussian and Eisenstein integers, which live in the complex plane and form square and triangular lattices respectively. This helped me understand their geometry in algebraic terms—because these lattice points are *algebraic integers*—comprising a Euclidean domain. This was an important paradigm-shift for me, as I was able to discover “prime curves”, and other number-theoretical aspects [18]. My fractal curve algorithm was revised so that all curves could be generated using operations on complex integers. As a

by-product, the parameter space for categorizing and discovering curves became smaller and more efficient.

I was asked later if I was absolutely sure that only these two lattices admit plane-filling curves [9]. This question made me wonder if the Gaussian and Eisenstein integers could be understood in a larger context. My answer came from Stefan Pautze, who developed a method for generating beautiful aperiodic tilings using *cyclotomic fields* with 5-fold symmetry, 7-fold symmetry, and others [10]. I asked Stefan if he thought it was possible to construct a fractal curve that conforms to the pentagon. He and I decided to set a challenge for each other: to come up with plane-filling curves that fill the regular pentagon. This paper presents a few of the results from that challenge.

Process

What are cyclotomic fields? Allow me to explain them in terms relevant to how they entered into my toolset. Figure 5(a) shows the roots of unity in the first 6 cyclotomic fields. The 2nd cyclotomic field has two units: -1 and 1. From these units, you can get any rational integer using sums and products. Similarly, from the units of the 4th cyclotomic field (1, i , -1, $-i$) you can get any Gaussian integer. And the units of field 6 can generate any Eisenstein integer. Taking inspiration from Stefan's technique, I can now use the units of fields 8, 10, and 12 (Figure 5(b)) to derive the parameters for generating curves. Note the irrational numbers $\sqrt{2}$, ϕ , and $\sqrt{3}$ that result from adding the first root of unity in each example with its conjugate. In the case of field 10, we get the parameters shown earlier in Figure 2. With this generalization, my palette grew to include the natural home of the pentagon...and (possibly) a much larger universe of plane-filling curves, in other cyclotomic fields.

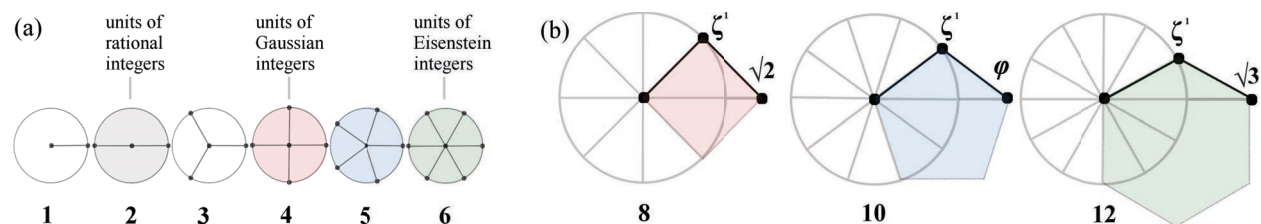


Figure 5: (a) The roots of unity in the first 6 cyclotomic fields; (b) using the 8th, 10th, and 12th cyclotomic units to derive parameters for generating plane-filling curves.

Here's how a graphic designer might describe a cyclotomic field: imagine having a rather odd drawing pen that can only draw one centimeter-long line segments (nothing shorter, nothing longer), and each segment is drawn in a tiny instant of time—a quick snap. Each segment is confined to a limited set of directions. In the 10th cyclotomic field, there are ten possible directions. Imagine drawing a wandering doodle on a drawing pad—without lifting the pen. Now imagine that this pen has controls allowing you to scale the length of each line by a power of ϕ .

For each segment you draw, you have to make two decisions: what direction, and what length? The two integer exponents for ζ and ϕ are your parameters. These are what I use to determine the segments of a generator, and the subsequent segments in its iterated fractal curve, using complex multiplication. Drawing with this pen manually would not be fun at all, so imagine having an algorithm that determines a sequence of cyclotomic integers, using a recursive function, and the pen takes care of the mundane task of drawing the integers as line segments. There are two additional boolean values used for each segment; they determine the transform applied to the child generator that replaces the segment in the next iteration: (1) a 180-degree rotation about the segment center (a 'flip'), and (2) a reflection about the segment.

Using a Penrose Tiling as a Guide

Ramachandrarao, Sinha, and Sanyal introduced a generator that forms the edges of a Penrose tiling when iterated to an arbitrary level of recursion [15], suggesting that Penrose tilings have a fractal nature. Penrose tilings provide a convenient framework as a guide for discovery (in this case, using variations on Robinson triangles [16]). One of my attempts at designing a set of generators is shown in Figure 6(a).

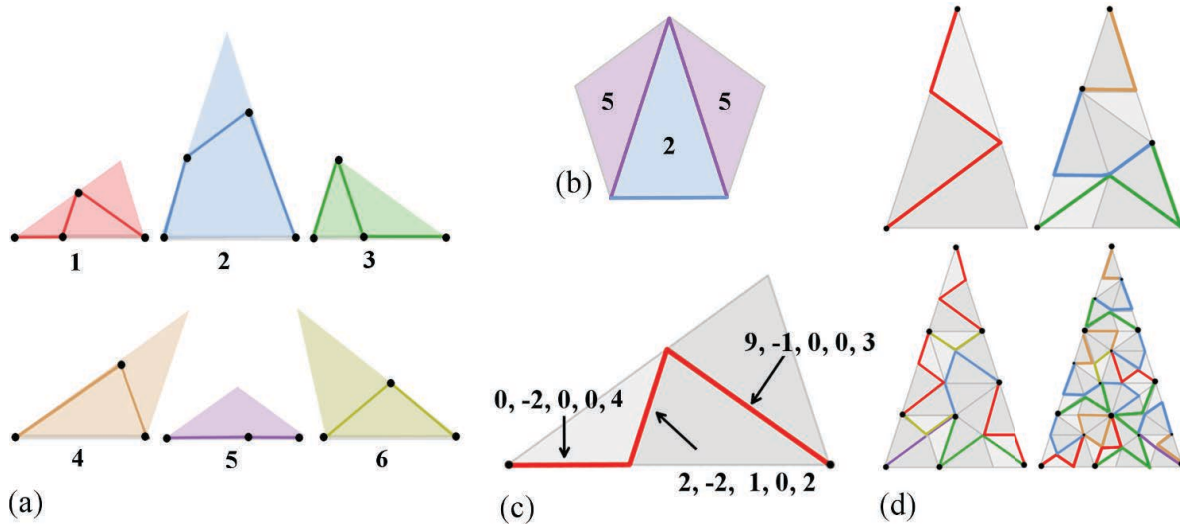


Figure 6: (a) 6 generators; (b) an initiator; (c) parameters; (d) iterations of a plane-filling curve.

In the new scheme, a recursive function references a small collection of generators that replace the segment in the curve, at every iteration. Thus, what determines a unique fractal curve is a set of generators, with one generator designated as the starting-generator. Each segment in each generator has 5 values (Figure 6(c)): an exponent for ζ , an exponent for φ , two boolean values that determine the transform, and the index to the generator that replaces it. (Note: the negative exponents for φ are responsible for *scaling-down* the child generator to fit on the segment it replaces, and so these segments are no longer strictly integers. I believe it is possible to make a variation of this technique that maintains integer integrity throughout the process: a subject for future consideration). Figure 6(d) shows the first three iterates of the generator in Figure 6(c). I'll leave it to the enthused reader to guess the parameters for the other generators (or better yet: to come up with a better solution). I combined three of the generators to make a closed pentagonal initiator (Figure 6(b)), the result of which is shown as Figure 1(d) (a smoothing filter is applied to the entire curve to separate overlapping segments, thus making it trivially self-avoiding). The result has similarities to Henle's pentagon (Figure 4b).

Results

By adding the appropriate tile to each segment in the curve (from the set of six colored Robinson triangles in Figure 6(a), but with a nicer color palette), this curve becomes a Penrose tiling. Because of the variety of generators used, this tiling has a rich, aperiodic pattern.

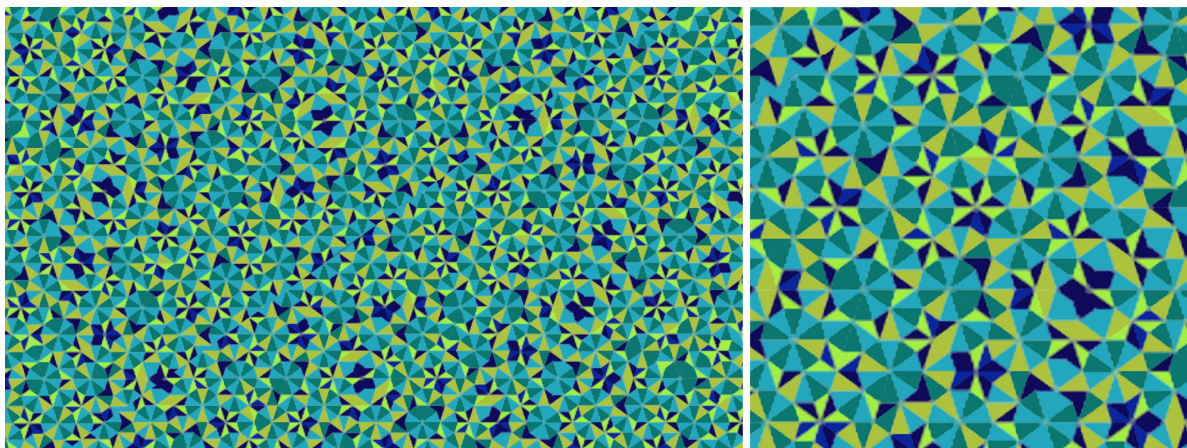


Figure 7: Penrose tiling (close-up at right) created by placing tiles on segments of a plane-filling curve.

Stefan Pautze came up with a more elegant pentagonal curve [11]. It requires only two generators. Figure 8 shows this curve with a smoothing filter applied. This curve has a wonderful feature: there are contiguous chunks of the curve that fill regular pentagons, which—naturally—repeat at multiple scales (powers of ϕ). This is made more apparent by highlighting pentagons using colors and filled-in regions. Note that every pentagon highlighted has a single curve passing through it. In two of the pentagons, the endpoints of their internal curves are shown with red arrows.

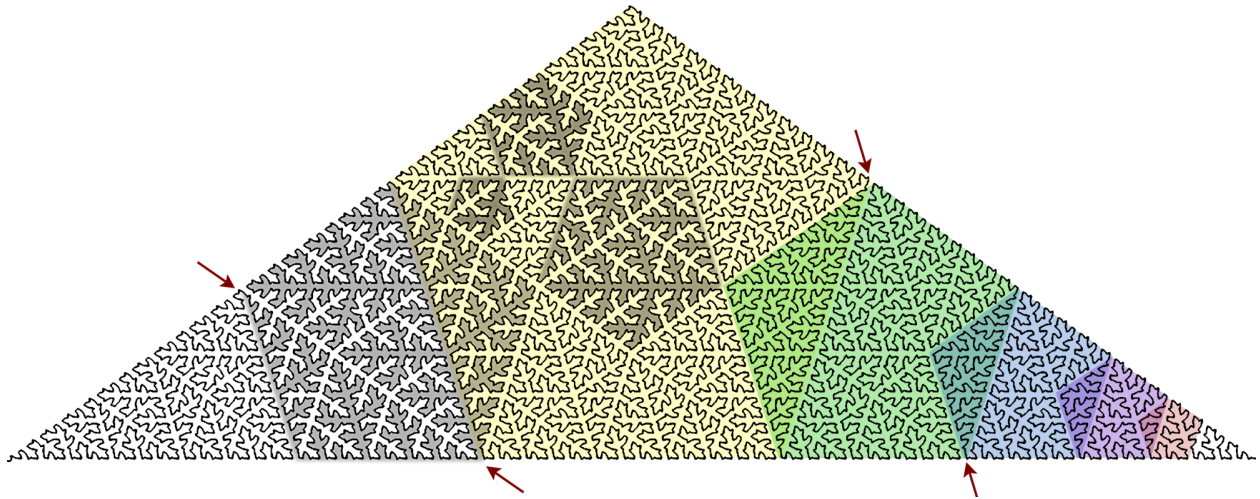


Figure 8: *Pautze curve enhanced with color and shading to emphasize multiple internal pentagons*

Albrecht Dürer explored progressively adding pentagons around a central pentagon [2]. The 6-pentagon generator can be used to build a self-similar fractal (Figure 9(a)). Repeated iteration resolves to a gasket. The same gasket can be achieved with a self-avoiding fractal curve, which I call the “Dürer Pentagon curve”. It requires two generators (Figure 9(b)).

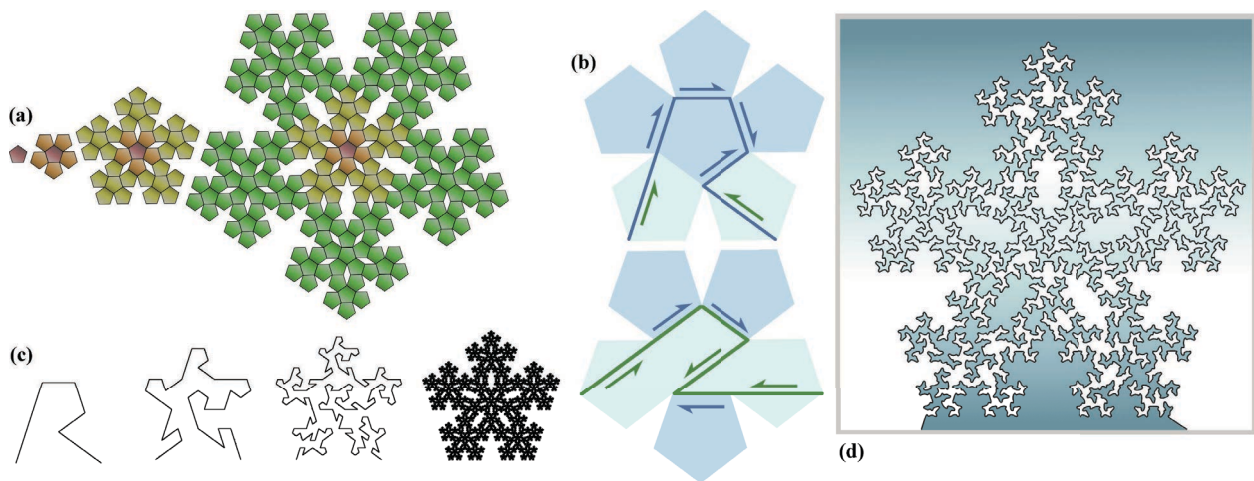


Figure 9: (a) *Iterations of the Dürer pentagon tiling;* (b) *generators for the Dürer pentagon curve;* (c) *the first few iterations of the curve;* (d) *a color-filled rendering of the curve.*

Figure 10 shows a variety of designs made using combinations of generators, transforms, tiles, and colors.

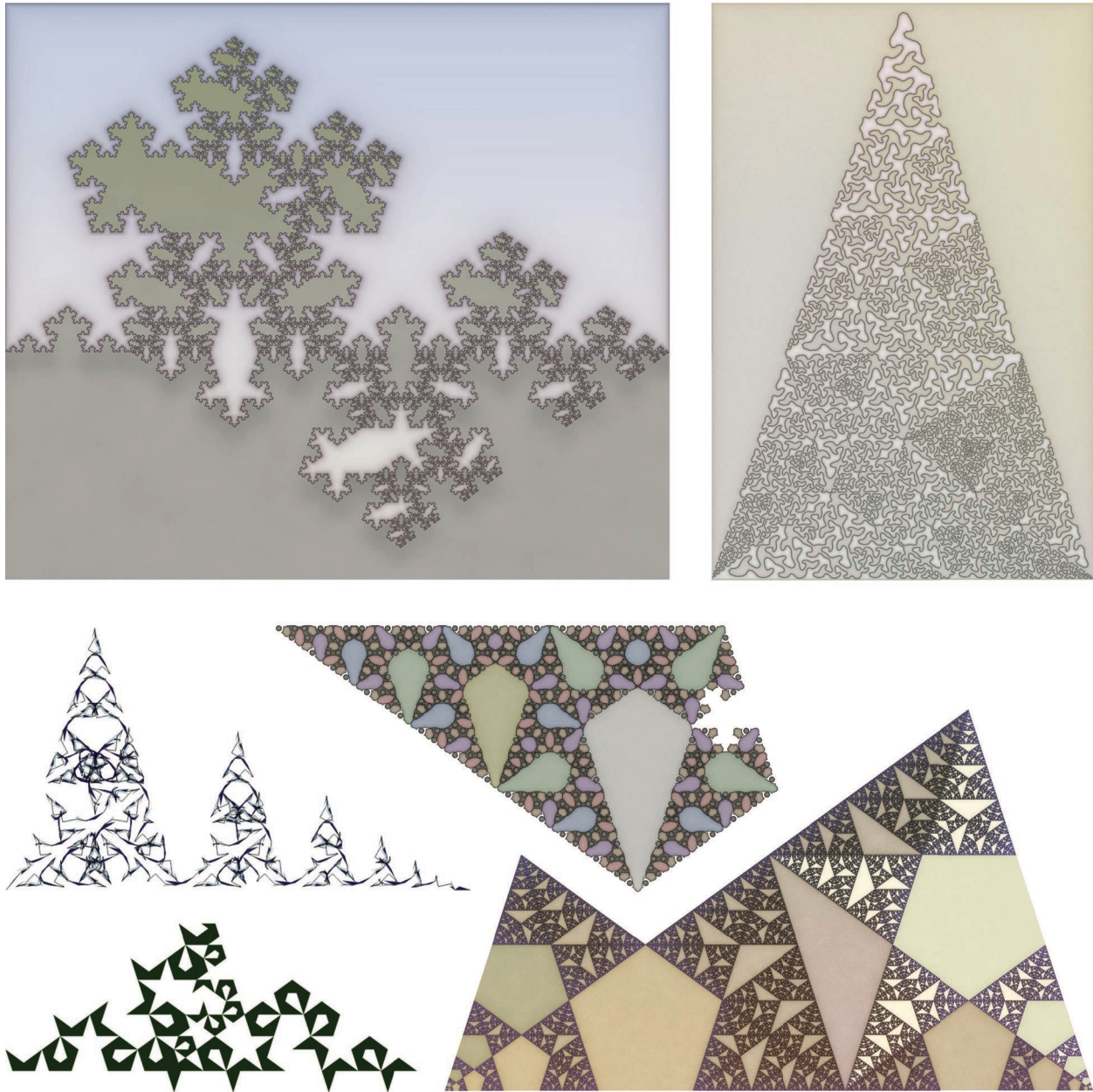


Figure 10: Selected examples showing a variety of designs, using color-filling, and various tilings.

Summary

The fact that pentagons do not tile the plane—the *trouble with five*—is an invitation to come up with creative alternative solutions, which date back to Kepler, Dürer, and the Islamic designers of the Medieval period. In this paper, I have described one approach; from the perspective of fractal curves. In the process of searching for plane-filling curves with five-fold rotational symmetry, some interesting designs have emerged, as well as a deeper appreciation for the mathematics of complex numbers, and in particular, the 10th cyclotomic field, where pentagons and pentagrams feel quite at home.

These curves could easily have been generated with standard turtle graphics techniques where segments are defined by real number angles and lengths. I would recommend that to anyone getting

started with fractal curves, and that was the case with my original learning process. But the progressive discovery of abstractions that eventually brought me to number theory have opened up an expanding universe of possibilities. This exploration is ongoing. Perhaps more mathematically and visually stimulating fractal curves will be discovered using this approach, giving rise to inspiration for new visual designs.

Acknowledgements

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