Generalizations of Truchet Tiles

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Abstract

Truchet tiles and their variants have long been studied for their mathematical and aesthetic interest. This work presents several generalizations to the traditional tiling rules, including using different tile shapes, more and different types of arcs, and combining tiling images into final artworks.

Truchet Tiles

Father Sebastien Truchet's original 18th-century tiles (1) were four squares split diagonally, with half black and half white. They are shown in Figure 1, along with an example pattern. Among his other pursuits, Truchet explored imagery created from patterns of the tiles. The set of infinite patterns that can be created is said to be a foundational idea in the encoding of graphics on computers. In 1987, C.S. Smith recognized Truchet's "graphical treatment of combinatorics" and presented his own variation, which are widely known today as "Truchet tiles." Smith used quarter-circle arcs in diagonal corners. Flipping one tile, either horizontally or vertically, produces the other. Because the arcs meet the sides of the square perpendicularly and at the centers of the sides, any orientation of the tiles in an edge-to-edge tessellation will allow the arcs to meet, forming smooth seamless blobs and loops. Figure 2 shows two examples, one emphasizing the arcs and the other highlighting the disjoint regions that they create. The tile boundaries are shown in gray. Many artist's have employed Smith's version of Truchet Tiles in their works; see for example Pickover (2).

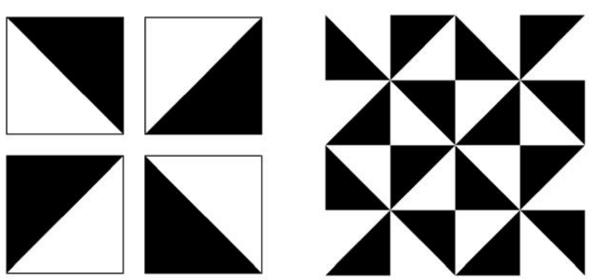


Figure 1: *Truchet's original tiles (left) and an example image (right).*

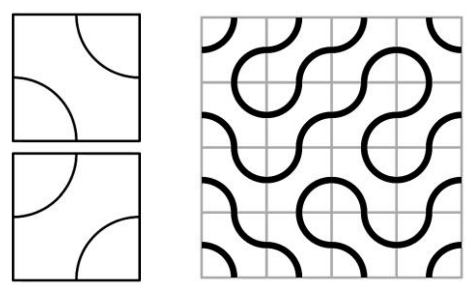


Figure 2: *Smith's variation of Truchet's tiles (left) and an example image (right).*

More Arcs and More Tile Shapes

Smith's work can easily be extended without changing much of the spirit. By adding another arc-termination point to each side (two points per side, dividing each side into thirds), each tile will now have four quarter-circle arcs. Figure 3 shows six possible tiles. (Notice that the last two tiles have straight lines as arcs. These are degenerate cases of circular arcs as the radius goes to infinity.) Given the asymmetry introduced by the various arc configurations, far more tiles can be generated by flipping and/or rotating these basic shapes. Figure 4 presents examples of random tessellations using these tiles.

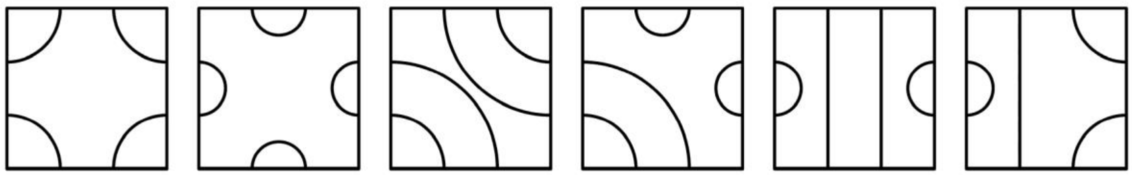


Figure 3: *Six square tiles with four arcs each.*

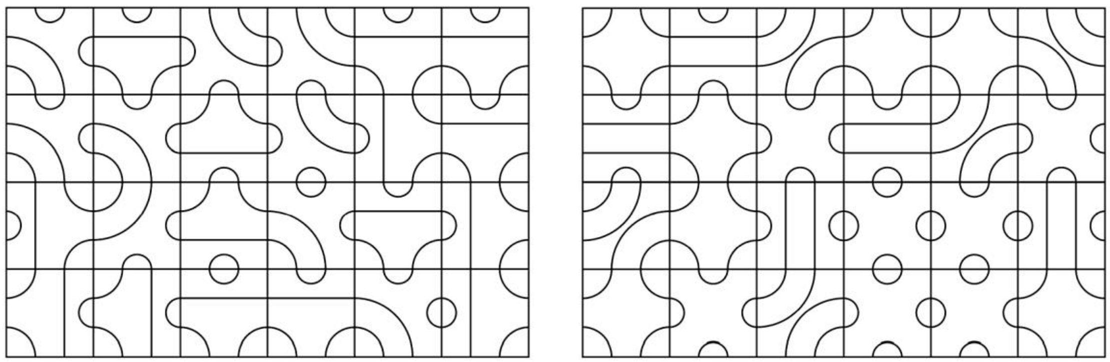


Figure 4: *Two example tilings made with four-arc square tiles.*

No doubt, the symmetry of the square and its ability to tile the plane figured heavily in the choices to use square tiles. However, there are other regular polygons that tile the plane, namely the equilateral triangle and the regular hexagon. Using the triangle as a tile shape presents a challenge. If we require arcs to terminate on an edge of the tile and to not intersect each other, then an even number of edge points are required, with two point per arc. The triangle has three sides, so a one-point-per-side tile like Smith's square tiles is not possible. However, two points per side is possible and the three such known tiles (up to rotation) are shown in Figure 5, along with a sample tiling.

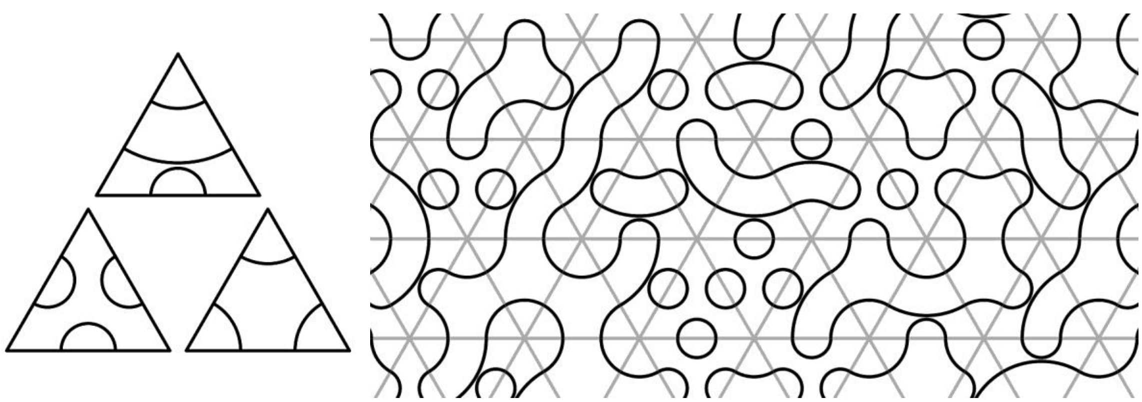
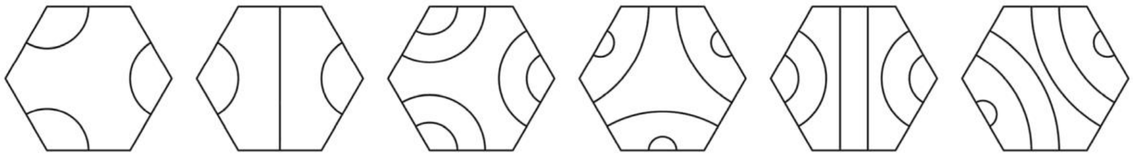
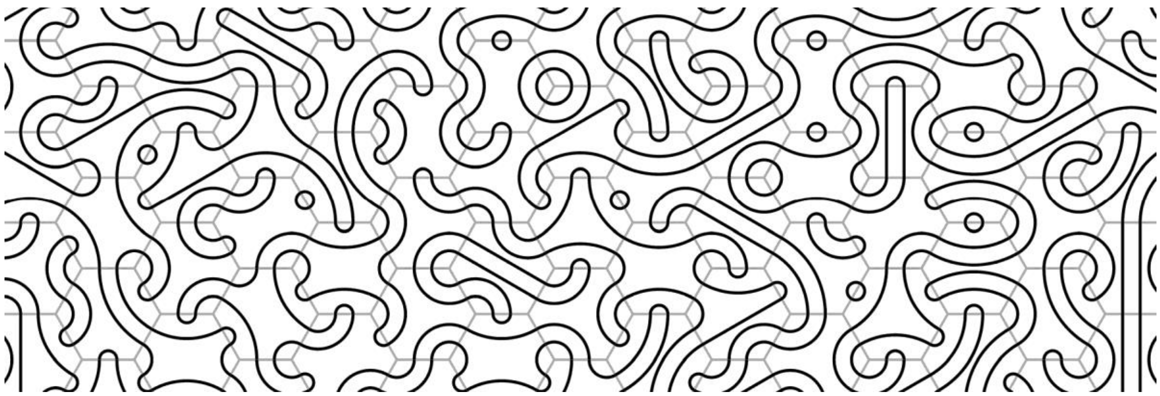


Figure 5: *Three two-arc triangle tiles (left) and an example tiling (right).*

Regular hexagons also tile the plane, and with an even number of sides, support any numbers of points per side. With more arcs than square tiles, hexagonal tiles present far more opportunities. Some examples are shown in Figure 6. In Figure 6(b), the tile boundaries are indicated by the lighter gray lines.



(a)



(b)

Figure 6: *Hexagonal tiles: (a) six sample tiles; (b) example tiling.*

Regular octagons don't tile the plane, but this provides a chance to add even more variety, by using Archimedean tessellations (3). As Baez (4) explains it, these are, "tilings by regular polygons where all the edges are the same length and every vertex looks alike." Having all the edges be the same length ensures that the arcs will meet. And since all the arcs are perpendicular to the edges, the paths will cross the tile edges smoothly. One such case is the truncated square tiling, in which each square shares every side with a regular octagon and each octagon shares half of its sides with squares and the other half with octagons. Of course, octagons have eight sides, allowing for even more complexity in the tile arcs. Figure 7 shows some example tiles and Figure 8(a) shows them in use with the truncated square tiling. The other two panels of Figure 8 illustrate possibilities with the snub square tiling (equilateral triangles and squares) and with the rhombitrihexagonal tiling (equilateral triangles, squares, and hexagons).

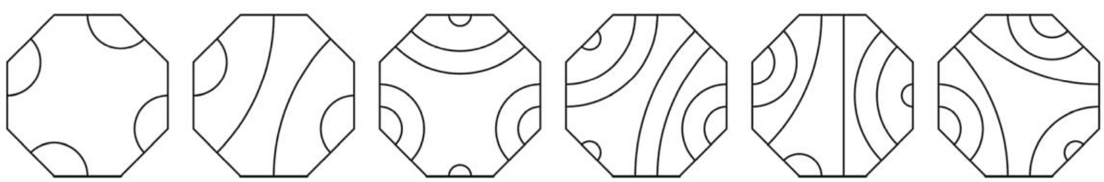


Figure 7: *Sample octagonal tiles.*

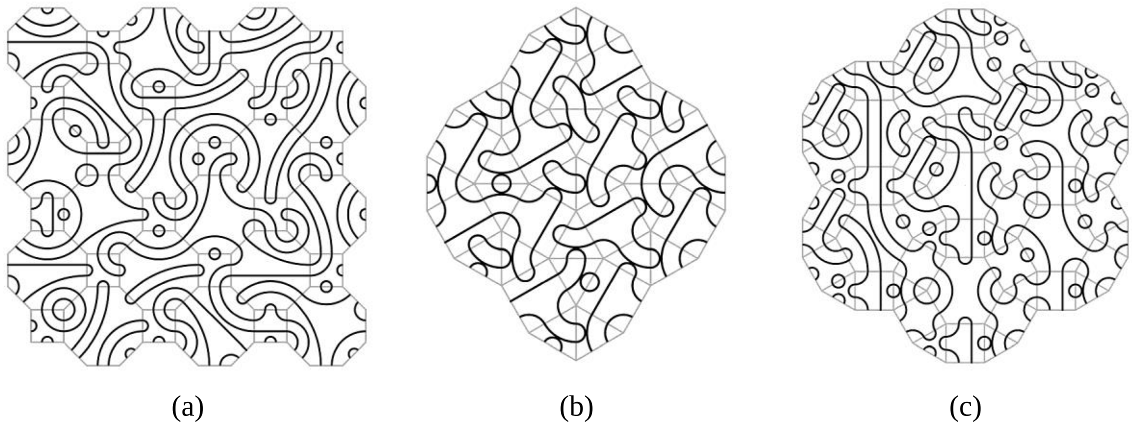


Figure 8: *Archimedean tilings: (a) truncated square; (b) snub square; (c) rhombitrihexagonal.*

To further generalize, I tried non-circular arcs. I felt that the requirement of having the arcs meet the tile edge perpendicularly needed to be maintained, so I opted to use Bézier curves (5) to define the arcs. Specifying the termination points of the arcs and their angles results in four constraints, so at least a cubic curve is required. However, using a quartic curves allows incorporating a free parameter (an extra point to define the arc). Adding a second free point (a quintic curve) provides for even more artistic freedom. Figure 9 shows three pairs of square tiles; the first in each pair uses circular arcs and the second, Bézier curves. The first two Bézier curves are fourth-order and the third is a fifth-order curve.

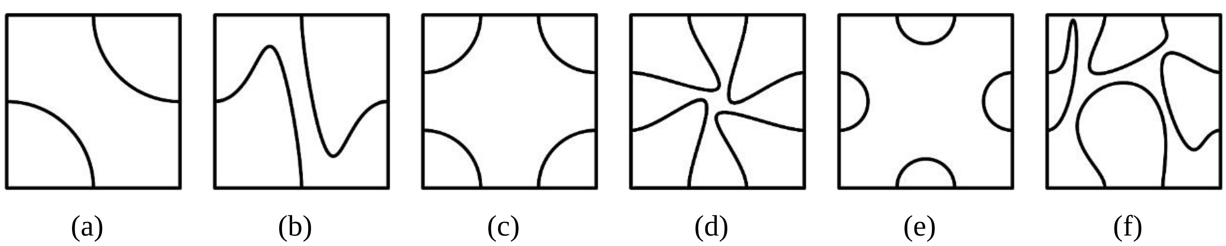
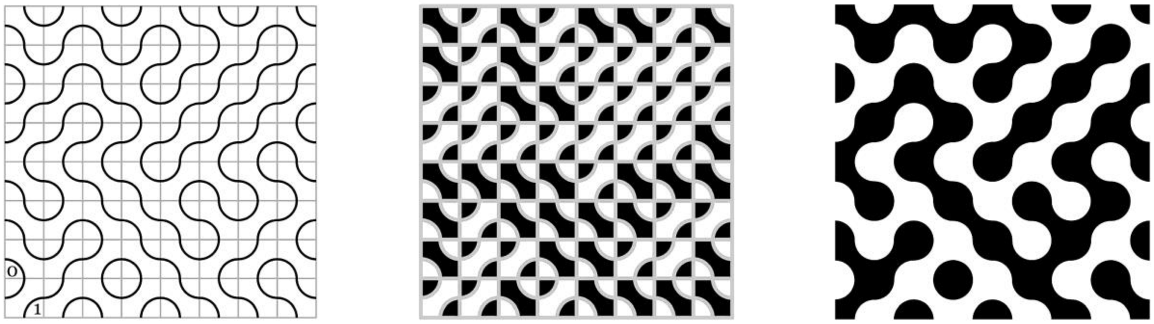


Figure 9: *Square tiles with circular and Bézier curve arcs: (a) two corner circular arcs; (b) two corner Bézier curve arcs; (c) four corner circular arcs; (d); four corner Bézier curve arcs; (e) four side circular arcs; (f) four side Bézier curve arcs*.

Using Tiles in Art

Frequently, Truchet tile images are created with random orientations of the tiles, arbitrarily choosing which tile (arc pattern) is placed where, while the tile shapes are arranged regularly. See Figures 1 – 5 for examples of this method. Alternatively, the tiles can be intentionally placed in patterns. Reimann (6) exploited the shapes of loops created by the Smith variants to encode text messages within Truchet tile images. In Figure 10(a), I used the two types of Smith tiles to represent binary digits 0 and 1 (see the labels as a guide to reading the digits). Then, the arrangement of the tiles can represent the binary expansion of a real number. In this case, the first 64 bits of $\sqrt{2}$ (approximately 1.414) are displayed.(Since $\sqrt{2}$ is an irrational number, its decimal expansion is infinitely long and could provide data for an infinite tiling.) Further, the 64 tiles are traversed in the path of the level 3 Hilbert curve (7), from the lower left corner, then up, right, down, right, right, up, etc. (1.011001...). Using other sets with more tiles can allow for encoding more digits, and thus, expressing numbers in other bases.

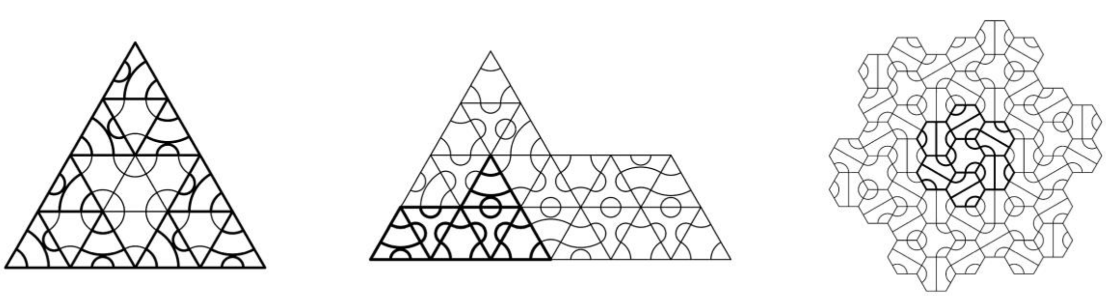


(a) (b) (c)

Figure 10: *Using Truchet tilings to encode bits of a binary number: (a) tile outlines and arcs; (b) alternate rendering coloring regions of the tiles; (c) alternate rendering revealing closed regions.*

Figure 10 also demonstrates different ways in which these tilings can be rendered, using the same tiles and arrangement. In Figure 10(a), the arcs are featured, while the tile boundaries are added as a guide to showing how the image is constructed. Conversely, in Figure 10(b), the tile boundaries are highlighted (as are the arcs), to prominently outline the various disjoint regions that the tiles establish. These regions are then colored alternately black or white, for hightened contrast. Finally, Figure 10(c) dispenses with the boudaries completely and the arcs are inferred as outlines of closed curves. The interiors of the curves are colored, emphasizing the various features that can be generated with (in this case) two simple types of tiles placed on a regular grid.

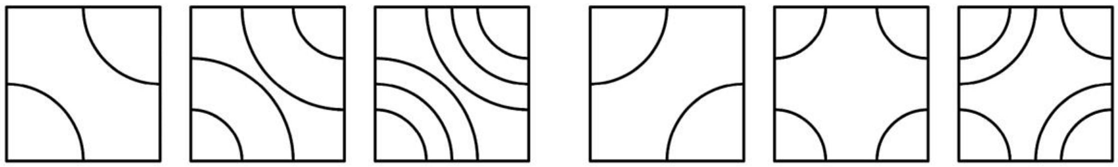
In this work, all the tiles being considered are convex polygons. This opens up the possibilities of combining tiles into meta-tiles, which then serve as the unit used to tessellate the plane. Figure 11 shows three examples. In Figure 11(a), triangular tiles form a low-level approximation to the Sierpiński triangle (8). Two types of tiles are used, one with two arcs centered at one corner and one arc centered on the opposing side (drawn with heavier lines) and one with one arc centered at each corner (lighter lines). Instead of removing triangles at each step, the processes suggested here is to build up the next step from three copies of the previous iteration, plus a new central triangle of the three-corner-arcs type. Figure 11(b) shows the "sphinx" tiling (9). In the lower left corner, a set of six triangular tiles is highlighted. This is the basic form of the sphinx. Adding three copies (rotated and/or flipped) yields the larger sphinx shown. Repeating this process shows that the sphinx tessellates the plane. In Figure 11(c), seven hexagonal tiles are highlighted in the center of the image. These tiles form a quasi-hexagonal meta-tile (of course, the meta-tile is not a hexagon as it has more than six sides). Repeating this with the meta-tile yields the 49-tile larger version, which can then be used to make a 343-tile version, etc. Square tiles can be used in a myriad ways to build up larger shapes (polyominoes), which can then be used to make more shapes.



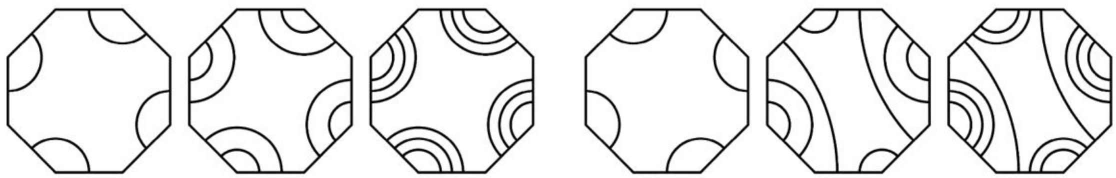
(a) (b) (c)

Figure 11: *Combining tiles into meta-tiles: (a) Sierpiński triangle; (b) sphinx; (c) hexagonal meta-tile.*

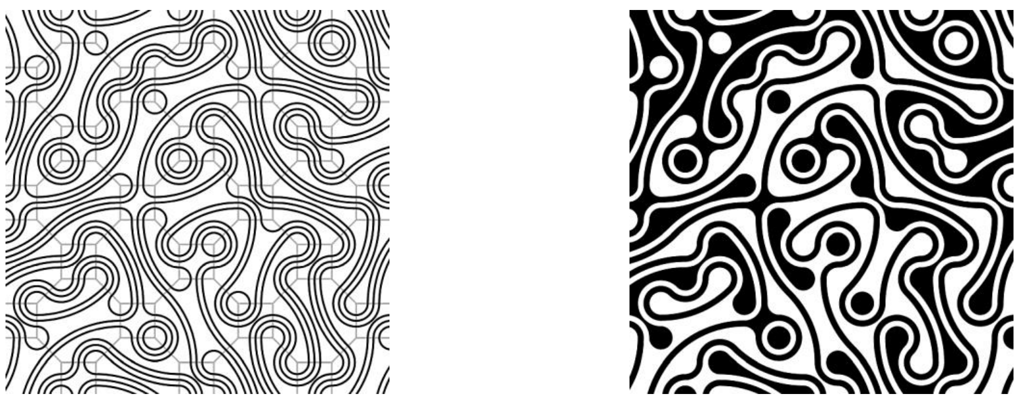
In some cases, it may be possible to composite multiple tiles of the same shape into new tiles with more arcs. Figure 12 shows four cases. The first two (Figure 12(a)) show examples of overlaying two square tiles, one with two arcs and the other with four, into tiles with six arcs. The other two (Figure 12(b)) show creating a new octagon tile from two others, one with four arcs and the other with eight. These new tiles are used with the truncated square tiling to create the images in Figure 12(c).



(a)



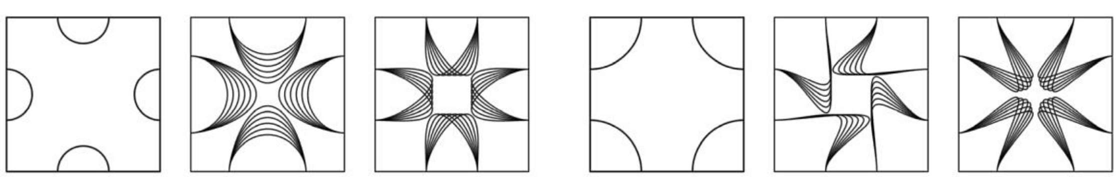
(b)



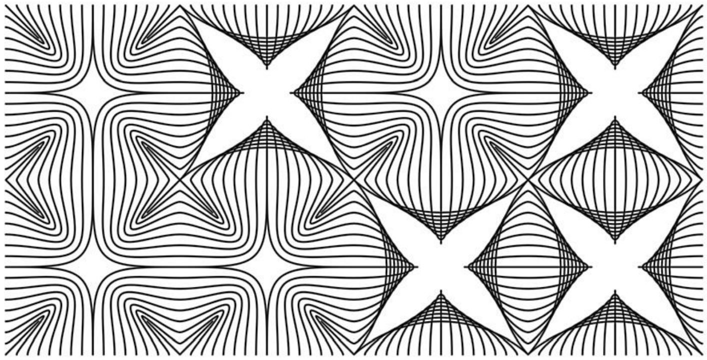
(c)

Figure 12: *Compositing multiple tiles: (a) two examples each with two square tiles; (b) two examples each with two octagonal tiles; (c) two renderings of an example tiling.*

Using Bézier curves, there are literally infinitely many arcs that can be drawn for any tile configuration. This presents an opportunity to create hybrid arcs by compositing together many individual arcs. The collection respects the spirit of having the arcs not intersect on the tile, even though the individual curves may not. Figure 13(a) presents two examples for each of two square tiles. In two sets of three images, first, the tile with single circular arcs is shown, then two corresponding cases of Bézier curve tiles, each with six overlaid arcs. Figure 13(b) shows how the hybrid tiles might look in practice.



(a)



(b)

Figure 13: *Using multiple Bezier curves as a single composite arc: (a) two examples with square tiles; (b) example tiling.*

Finally, I present some examples of my own pieces, using the ideas described herein. Figure 14, "Truchet Hexagon 2B," is a combination of two 37-tile tessellations with six-arc tiles (such as those in Figure 6). Each was rendered in the manner of Figure 12(c), then the difference between became the final image. Figure 15 is "Truchet Triangle-Hexagon 2B," a composition of two Archimedean tilings of equilateral triangles and hexagons. Each 216-tile tessellation was rendered in the manner of Figure 10(a), just showing the arcs. The two images were overlaid and then the regions defined by the combined curves were colored. In the original, the solid gray regions are red. Figure 16, "Woman with Guitar," combines two random 100-tile truncated square tilings (see Figure 8(a)), each colored in the manner of Figure 10(c), but with different colors. In the original, one is blue and white and the other is orange and white. Figure 17, "Hummingbird," is a study of variations in two-arc square tiles. Each of the 16 tiles is rendered using 21 Bézier curves. Here, the individual curves and the compound arcs they form, were allowed to intersect, for aesthetic interest. In the original, the arcs on the tile in the second row from the bottom, rightmost column, are red.

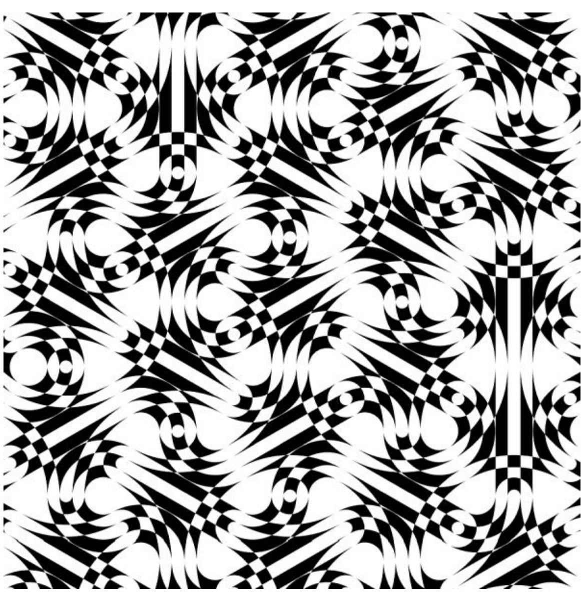


Figure 14: *"Truchet Hexagon 2B," by the author.*

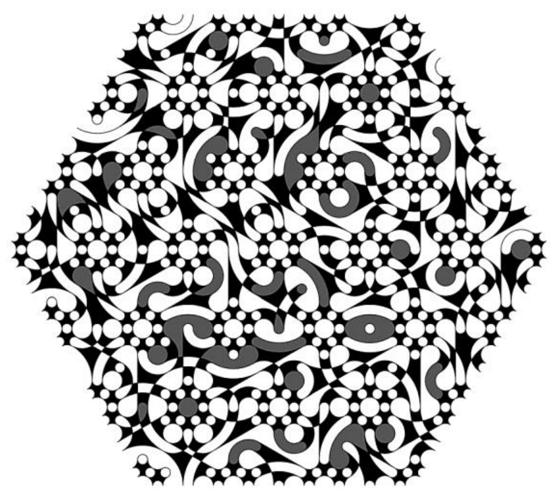


Figure 15: *"Truchet Triangle Hexagon 2B," by the author.*

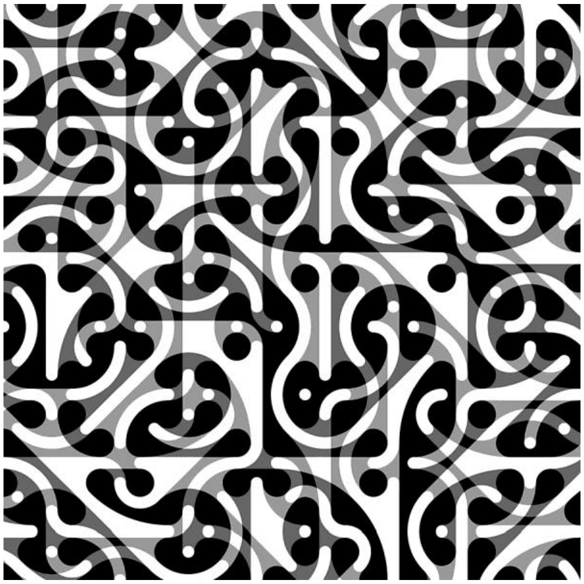


Figure 16: *"Woman with Guitar, by the author.*

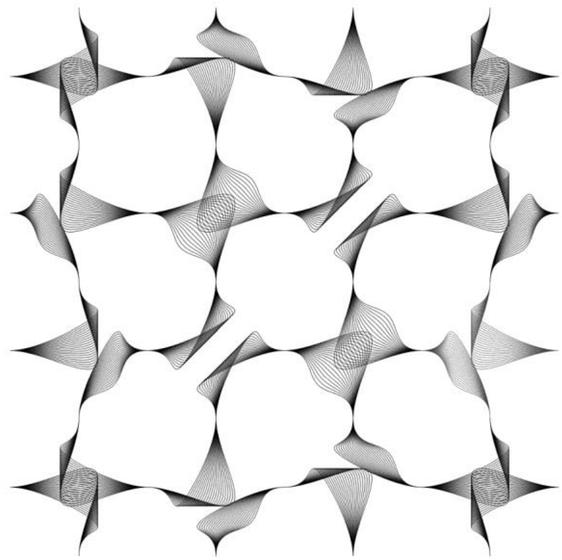


Figure 17: *"Hummingbirds," by the author.*

Summary and Conclusions

The tiling system invented by Truchet, and modified by Smith, is a very simple one with many complex ramifications, even with only a few tiles. By expanding the parameters to include more tile shapes, more arcs per tile, combinations of shapes, and compositing multiple tile image layers into a final image, the possibilities are literally endless. Here, we have considered other regular polygons beyond squares, the use of single and multiple polygon shapes in tiling the plane, expanding the number of arcs, and moving from circular arcs to more complex curves. In addition, aesthetic considerations for creating artworks from these tilings have been presented, along with several examples.

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