# Wallpaper Patterns from Nonplanar Chain Mail Links

Frank A. Farris

Department of Mathematics and Computer Science, Santa Clara University; ffarris@scu.edu

## Abstract

We explain techniques for creating mathematical chain mail, by which we mean patterns made from identical linked ring shapes. Most chain mail currently in existence uses circular, or at least planar, rings which are linked in a variety of ways. By contrast, we show how a small nonplanar "wiggle" in the shapes permits a marvelous variety of woven patterns. We give formulas general enough to capture every possible periodic chain mail pattern, but admit that the issue of weaving and self-avoidance of the rings is a decidedly empirical one. We show photographs of 3D-printed chain mail as well as virtual images, created and rendered in *Rhino* using the *Grasshopper* plugin. We mention color symmetry in chain mail, as well as possibilities for covering surfaces in mail.

## Introduction

Creating strong fabrics from linked rings, known variously as *chain mail* or *chainmaille* is almost as old as the human use of iron [6]. There seems to be a lively culture of preserving and expanding the tradition [7]. The physical realities of working with metal have led designers to work almost exclusively with circular, or at least planar, rings. By allowing the links just a bit of non-planar wiggle, we produce a wide variety of new doubly periodic patterns. We give a method for constructing chain mail of any desired wallpaper symmetry.

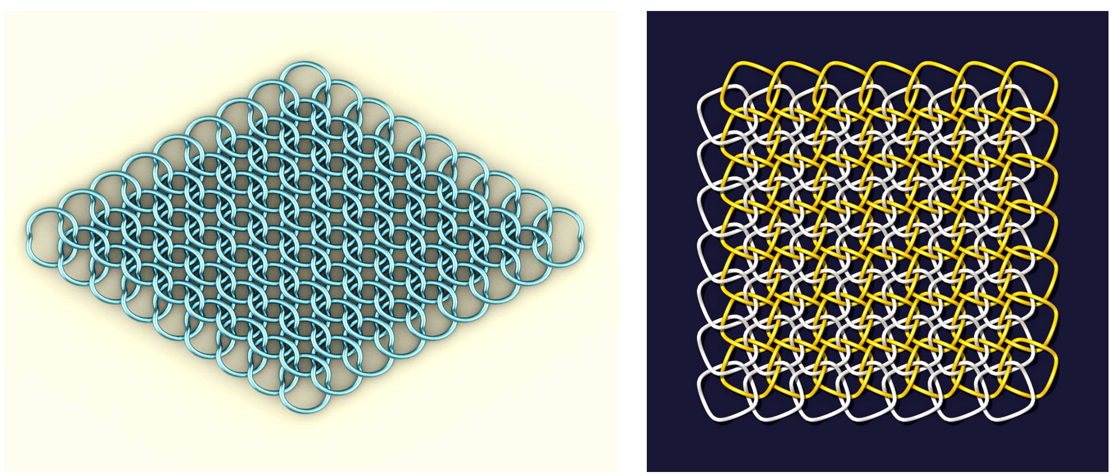

**Figure 1:** *Two samples of mathematical chain mail. a) Centered cell. b)* 4*-fold symmetry.*

Unlike existing treatments, which provide diagrams to explain how to link circles or other shapes into a pattern [6, 4, 8], we give mathematical formulas for the rings themselves. This allows much greater flexibility, at the cost of producing artifacts that are more easily produced by a 3D printer than a blacksmith.

The method is based on Fourier series for curves in a plane, with an additional Fourier series used to control up-and-down variation from that plane. After choosing a single link shape, we propagate it using the translations for whichever symmetry type we hope to achieve. The method is a combination of theoretical and empirical: The formulas we offer guarantee the desired symmetry of the links and the overall pattern;

whether or not the links intersect one another is separate issue. Still, as we will explain, the chain mail patterns seem to pop into being after some patient experimentation.

This inquiry might have been possible back in the earliest days when Fourier series were known. However, two things make the examples in this paper distinctively 21st-century artifacts: They were designed in *Rhino*, 3D modeling software with its *Grasshopper* plug-in for creating shapes from mathematical formulas. And the physical examples were 3D-printed, using a dual-head Ultimaker II printer with soluble support material.

In addition to producing samples of chain mail wallpaper patterns, we explain how *wallpaper symmetry* is not quite the right word here; the symmetry groups of our 3-dimensional examples are in fact called the *layer groups*. Still, our reference to wallpaper symmetry is unlikely to be misunderstood. We show how to achieve color symmetry of various types, using links of different materials. We also point to future work, where we create surfaces in space from chain mail links of variable size.

## Shaping a Single Link

On the right in Figure 1, we can see that each link essentially has the shape of a rounded square, which is one ingredient for the evident 4-fold symmetry of the pattern as a whole. In this section, we explain how to create curves with rotational symmetry, as the first step in making a chain mail pattern. For the sake of a self-contained description, we repeat some material that has appeared elsewhere [1, 2].

For convenience, we use a single complex number to denote a point in the $x$-$y$ plane and write a generic curve as a parametrized succession of complex numbers:

$$\gamma(t) = x(t) + i y(t).$$

A circular link would be parametrized simply by $\gamma(t) = e^{it} = \cos(t) + i\sin(t)$ for $0 < t < 2\pi$. For simplicity we will maintain this same parameter range throughout.

The main result of Fourier series is that any sufficiently smooth curve can be written as a Fourier series in the form

$$\gamma(t) = \sum a_n e^{int}, \tag{1}$$

where the coefficients $a_n$ are any complex numbers, and the sum is taken, potentially, over all integer values of $n$. The integer $n$ is called the *frequency* of the term. In our examples, the sum will be finite, as only a few nonzero coefficients give us a very large design space in which to look for chain mail links.

To ensure $k$-fold rotational symmetry in particular curve $\gamma(t)$, we could require it to satisfy the functional equation

$$\gamma(t + 2\pi/k) = e^{2\pi i/k}\gamma(t),$$

which means that advancing the time variable by a fraction $1/k$ of the full domain achieves the same effect as rotating the curve through an angle equal to $1/k$ of a full revolution. (There are slightly more general symmetry conditions [1, 2] for curves, but this one meets our needs very well.)

Computation shows that a curve given by a Fourier series as in (1) will satisfy the given condition for $k$-fold symmetry if and only if $a_n = 0$ except when $n \equiv 1 \mod k$. In order words, if we wish to create a curve with $k$-fold rotational symmetry, we should combine terms where the frequency $n$ is one more than a multiple of $k$.

For example, the rings on the right in Figure 1 are based on profile curves of the form

$$e^{it} + a_{-3}e^{-3t},$$

with the frequency $-3$ being one more than $-4$. In other examples, we use three or more terms, but a surprising variety of shapes can be achieved within quite a limited range of terms. After all, each additional term with a nonzero complex coefficient adds two more real dimensions to the design space of curves.

As an aside, if we wanted to give the rings a perfectly square profile curve, we would need to include infinitely many terms. Even so, stopping with 30 or so terms produces the appearance of quite a sharp square corner [1].

**Wiggling the band up and down** To deform our curve out of the design plane, we add a third component, which will be a real number $z(t)$ for each $t$. The condition for $k$-fold rotational symmetry of the band about the $z$-axis is $z(t + 2\pi/k) = z(t)$, in order that the band repeat the up/down wiggle over each fraction $1/k$ of its domain. If we express $z(t)$ as a Fourier series

$$z(t) = a_0 + \sum_{n>0} a_n \cos(nt) + \sum_{n>0} b_n \sin(nt),$$

then symmetry will be achieved if and only if $a_n = b_n = 0$ except when $n \equiv 0 \mod k$. For the right-hand image in Figure 1, we used the simple formula $b_4 \sin(4t)$. It surprised me that nothing more tricky was required to produce the wonderful weaving in the figure.

## Translating Copies and Searching the Design Space

We have established a formula, with as many free parameters as we might wish, that is guaranteed to produce a curve with whichever rotational symmetry we desire. The next step is to spread copies of one curve in a pattern likely to give us good chain mail.

Although any physical example of chain mail will be finite, it is simplest to talk about filling the plane with links, without a predetermined stopping point. Therefore, we consider a *lattice* of translation vectors, which is a set of integer multiples of two given vectors. In symbols, a lattice looks like this: $\{a\vec{v}_1 + b\vec{v}_2 \mid a, b \in \mathbb{Z}\}$.

For the chain mail with square symmetry on the right in Figure 1, we take the generating vectors to be the unit coordinate vectors. However, that specific example involves a twist that we will discuss later. Instead, let our first example of translation be the vectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = \left(-1/2, \sqrt{3}/2\right)$, which generate a grid of equilateral rhombi with 3-fold rotational symmetry.

As we explain in a later section, starting with a single ring with 3-fold rotational symmetry and translating infinitely many copies of it by the vectors in this particular lattice will produce a pattern with wallpaper symmetry. Before thinking about wallpaper groups, let us continue with some practical matters.

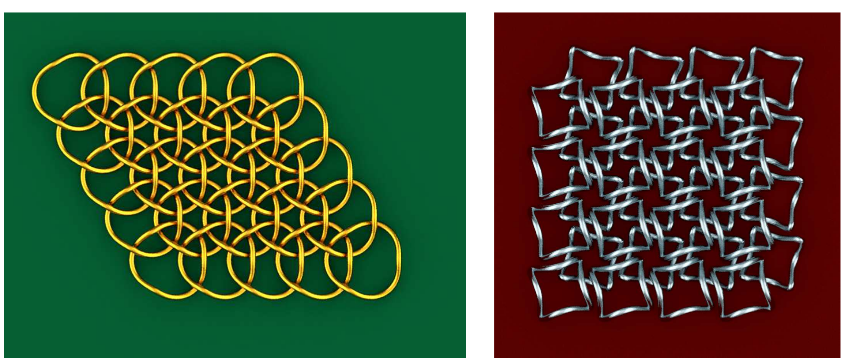

**Figure 2:** *a) Chain mail with 3-fold rotational symmetry. b) A funny failure.*

To make the pattern in Figure 2a, I used the recipes in the previous section to create the curve

$$\gamma(t) = \left(\mathrm{Re}\left(e^{it} + a_4 e^{4it}\right), \mathrm{Im}\left(e^{it} + a_4 e^{4it}\right), r_z \cos(3t)\right), \tag{2}$$

where $r_z$ and $a_4$ are *real* numbers. These choices give the links the mirror symmetry present in the pattern. We explain the many possibilities for additional symmetries in wallpaper in the next section.

Here is where the process becomes more experimental than mathematical. Readers who wish to try this for themselves will need software capable of displaying a collection of curves whose parameters can be controlled with sliders. I have used both *Maple* and *Rhino* with *Grasshopper*, with *Rhino* being the more flexible choice. Beyond displaying a collection of curves, one needs a way to create tubes (called *pipes* in Grasshopper) of circular or elliptical cross-section around those curves.

The visual programming interface provided in *Grasshopper* allows us to create these things and see them displayed immediately in the *Rhino* display window. I was quite surprised when I tried this and found that by adjusting the many parameters, I quickly found a configuration like that in Figure 2a. This was not an arduous process of forcing the links to interweave, but a matter of knowing what formulas create symmetry and then searching the parameter space for nicely woven patterns. Never in my mathematical career have I felt so strongly that these things wanted to be found.

While the weaving observed in these patterns might seem very special, there are theoretical reasons to suggest that they are perhaps not so uncommon. If we first think about the curves that lie at the centers of these tubes, we realize that, should we find intersections among neighboring curves, we could remove those intersections by wiggling the parameters just slightly. Topologists would say that the non-intersection of neighboring curves amounts to an *open* condition; probabilists might say that intersections occur with probability zero. Failing to intersect is the generic behavior here. Introducing the tubes around the curves is, of course, likely to produce intersections. However, given the open nature of the non-intersection condition, there is always a sufficiently small tube radius that would avoid intersection. This suggests that for any of the pattern types, some choice of parameters will produce non-intersecting links. Finding them is another matter and that's where experimentation comes in.

Figure 2b shows a funny failure of this method. As I varied the parameters to find this one, I focused more on the aesthetic appeal of the pattern and forgot to check to see whether the rings actually linked with one another. This chain mail is just a pretty stack of rings.

## Wallpaper Symmetry

The familiar theory of planar wallpaper patterns does not quite apply to the 3-dimensional chain mail samples we have constructed: Wallpaper groups [1, 3] act on the plane, but our chain mail patterns live in 3-space. Fortunately, a complete theory of how to extend wallpaper symmetry to three dimensions is well known. Before explaining the general situation, we highlight a pair of examples.

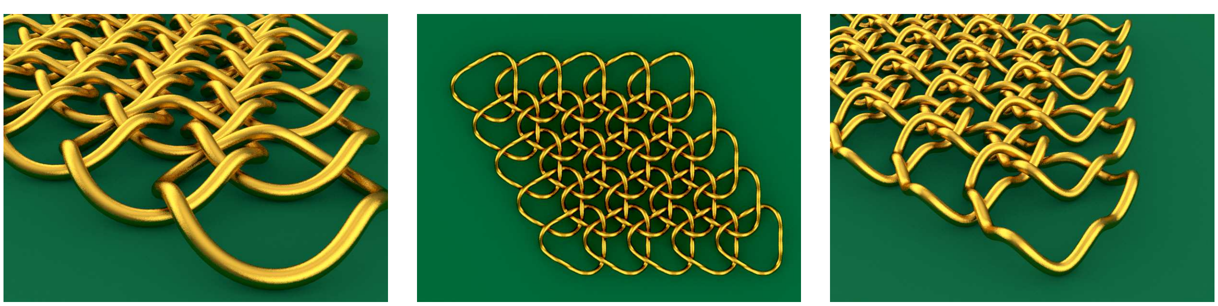

**Figure 3:** *(a) Close-up of Figure 2a to show mirror symmetry. (b) A similar pattern, but with flip symmetry. (c) Close-up to show flip symmetry.*

To make the chain mail Figure 2a, we constructed the planar base curve to have mirror symmetry across the $x$-axis, a reflection we call $\sigma_x$. This transformation of the plane, together with a 3-fold rotation about the origin, which we name $\rho_3$, and translation along the lattice vector $v_1$, are enough to generate a group of transformations of the plane isomorphic to the wallpaper group variously called p31m and 3*3. Extending each of these transformations trivially to serve as a transformation of space gives us one way to represent p31m as acting on space. Figure 3a shows a close-up of the pattern in Figure 2a, viewed roughly along the axis of mirror symmetry.

Instead of creating mirror symmetry about the $x$-axis, we could look for a pattern invariant under a half-turn about the $x$-axis, which we will call *flip* symmetry. The group generated by $\rho_3$, this flip, and translation along $v_1$, when restricted to the plane, is isomorphic to the planar group p31m, but a 3D pattern invariant under this group has a different appearance, as well, it turns out, as different material properties.

One example of chain mail with this kind of symmetry appears in Figure 3b, along with a close-up in Figure 3c. The general condition for this flip symmetry entails using real coefficients in the planar curve, just as in equation (2), but using sines instead of cosines for the $z$-component. In fact, for this example, I could not find a good example in the small design space with only one trigonometric term, so I tried $a_1 \sin(3t) + a_2 \sin(6t)$ and found the pattern shown.

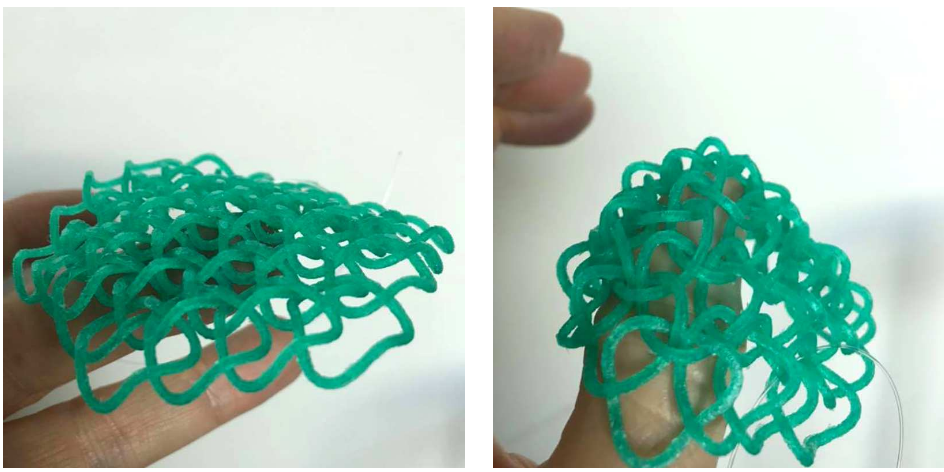

**Figure 4:** *The pattern with mirror symmetry from Figure 3 exhibits vastly different material properties on its two sides.*

**Material Properties** When these designs appeared only on the screen in *Rhino*, it was impossible to guess how they might behave in the real world. Printing the pair of examples from Figures 2a and 3 revealed an interesting difference: The sample with flip symmetry exhibited an equal tendency to bend whichever side was up. After all, it was designed with this symmetry. Something surprising happened when I held the sample with mirror symmetry. When supported on one side, it bent very little as in Figure 4a; turning it over led to the nice draping in Figure 4b! It makes sense that the shape without flip symmetry would be inhomogeneous in this manner.

**General Wallpaper Symmetry—The Layer Groups** When I embarked on this inquiry into chain mail, I imagined I would quickly make chain mail with every possible wallpaper symmetry. I did not realize the magnitude of this task. When one considers every possible way to extend one of the 17 wallpaper groups to a space group, it turns out that the possibilities are far greater than 17. Groups like these are called *subperiodic groups* because they are groups of isometries of 3-space whose translational subgroup is generated by fewer than three lattice vectors. The ones with a 2-dimensional lattice of translations are called *layer groups* and there are 80 of them [5].

In the 3-fold example, we saw two different ways to extend the action of the planar group p31m to space. In the first, we interpreted the generating mirror symmetry as just that: a mirror reflection in space; in the second, we interpreted it as a flip. In the first case, we constructed a pattern invariant under an indirect (orientation-reversing) symmetry of space; in the second, the pattern was invariant under a direct symmetry. Counting the possibilities amounts to going through the 17 wallpaper groups and counting the ways to extend the planar action of each group generator to an action on space, typically finding two possibilities: one direct and one indirect. (In more concrete terms, if you have a $2 \times 2$ matrix for an isometry, you can expand it to a $3 \times 3$ isometry matrix by padding with 0s and using either a $+1$ or a $-1$ in the bottom right corner.)

There isn't space here to walk through such a count and certainly no space to print 80 examples. Let us be content with one more pair of examples. In Figure 5a, we see a chain mail pattern invariant under a group generated by independent translations and a single rotation through 180° about the $z$-axis. The wallpaper group so generated is called p2, and this seems to be the natural way to extend that group of isometries to a group of isometries of space.

However, one could also extend that rotation, $(x, y) \to (-x, -y)$, to the point reflection, $(x, y, z) \to (-x, -y, -z)$. The pattern in Figure 5b has this symmetry. We invite readers to extend these ideas and perhaps create their own sampler of 80 types of chain mail, puzzling through the various systems of notation [5]. It is possible that some layer types cannot be realized in chain mail; I don't see any obstruction, but the most satisfying proof would come by constructing examples.

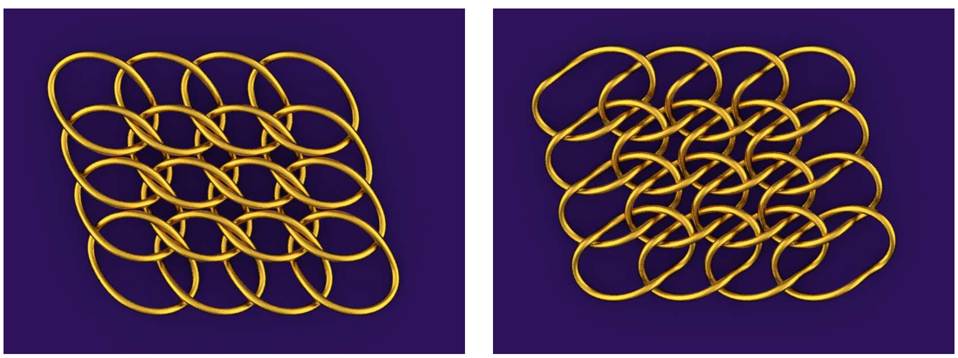

**Figure 5:** *Patterns illustrate two different layer groups that extend wallpaper group p2.*

Actually, the count of 80 groups may be too large for chain mail considerations, as it considers both *orthogonal* and *inclined* extensions. Since chain mail remains rather close to the translational plane, it may not matter much whether the transverse direction is perpendicular to the translational plane or not.

**Color Symmetry** We now discuss the color symmetry shown in Figure 1b, which allows us also to explain one more trick that seems to be necessary for producing certain pattern types. It is not quite true that one can always shape a single ring and propagate it out to a wallpaper pattern; sometimes a pair of links is required.

Consider the example of p4g, a wallpaper group generated by $\rho_4$, the 90° counterclockwise rotation of the plane about the origin, and $\sigma_d$, reflection about the line $x + y = 1/2$. It is a good exercise to verify that these two transformations can be combined to produce translations that generate the integer lattice.

To create Figure 1b, I first used the Fourier recipe to make a ring with 4-fold symmetry, then flipped a copy of it by applying $\sigma_d$ to create a pair of symmetric rings. (In light of the previous section, I could return and make that second copy a reflected one, rather than a flipped one, but the weaving would require substantial changes. In fact, this could be a type not realizable in nonintersecting chain mail links.) The pair of rings was then propagated using translations.

Figure 6 is a screen shot of the portion of the *Grasshopper* window I used to design this pattern. In

*Grasshopper*, the flow of the program is controlled by boxes connected by wires. Instead of lines of code, we see program elements with connections that establish desired operations. The three large boxes compute the $x$, $y$, and $z$ coordinates for the curve at the heart of a single link. They depend on the same time variable $t$, which flows into all three boxes from the left. Sliders control the parameters, which the reader may wish to connect to equations (1) and (2), adapting the latter for 4-fold symmetry.

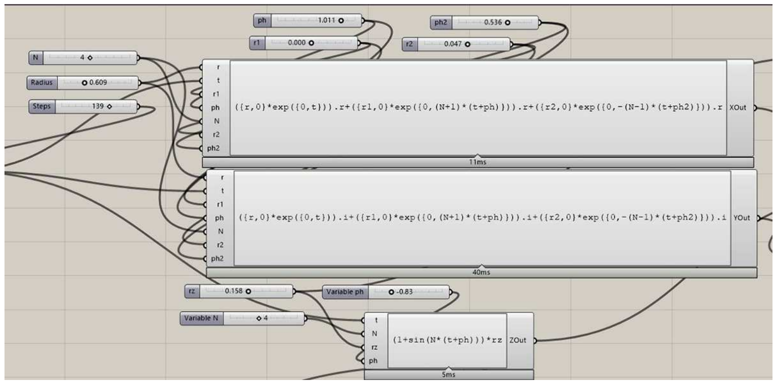


**Figure 6:** *A Grasshopper screenshot showing sliders to control the shape of a single link in Figure 1b.*

What has this to do with color symmetry? Since the primary and flipped rings were created separately, I could propagate each separately by the translations and save the sets to two different layers. Assigning different materials to the two layers, I achieved the 2-color effect in Figure 1b. The same can be done for any of the 2-color pattern types [1].

## Baskets, Spirals, and Conformal Chain Mail

The tools used to make copies of a single link of chain mail and move them across the plane apply equally well to moving links to cover a surface. It seems to me that the nicest way to do this is to find a coordinate system that is *conformal* to the Euclidean plane. In order words, we would like a parametrization of the surface by the map $\vec{x}(u, v)$ so that the basis for the tangent plane, $\vec{x}_u$ and $\vec{x}_v$, consists of orthogonal vectors of the same length as one another. This is certainly the case for ordinary spherical coordinates. Theoretically, such a coordinate system is always possible locally, though it may be difficult to find explicitly.

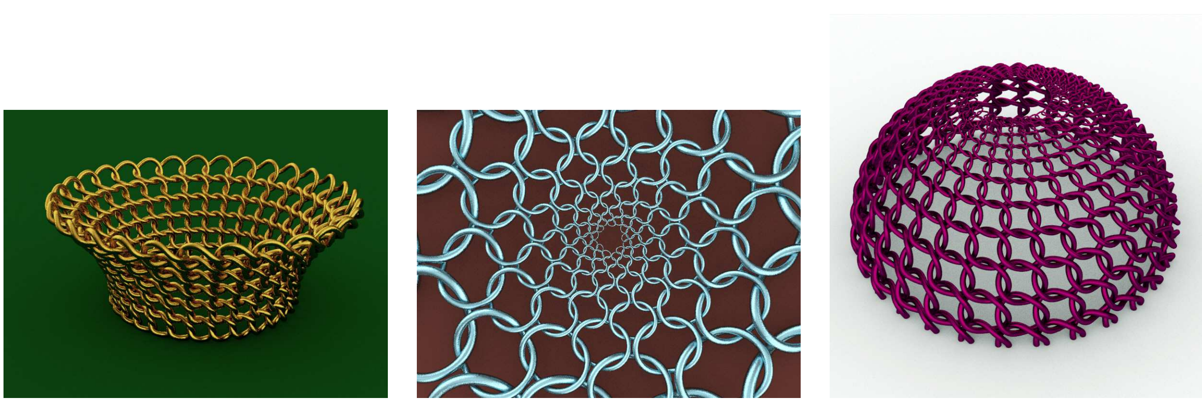

**Figure 7:** *(a) A chain mail basket based on the catenoid. (b) A Fibonacci spiral in chain mail. (c) Chain mail for the sphere.*

*Grasshopper* has a component that allows us to move a pipe to a location in space by specifying coordinate vectors for its base plane. If those coordinate vectors are stretched, the pipe stretches in proportion. As long as those two coordinate vectors have the same length as one another, the shape of the pipe will not be distorted.

Figure 7a shows this method applied to place a chain mail pattern on the catenoid, whose standard coordinate system conveniently has the right conformal property. The growth in size of the links toward the top is subtle, but this is how the same number of links can span both the top and bottom of this nice basket shape. Future collaboration with ceramic artist Timea Tihanyi is planned to produce objects similar to this one, printed with a ceramic 3D printer and fired.

The complex exponential function is a well-known conformal map from the plane to itself. The example in Figure 7b uses the complex exponential in an homage to John Edmark, whose "*Blooms*" project is well known. A special Fibonacci property of the grid allows a blooming effect when the design is animated. (An animated GIF is available as a supplement to this article.)

Finally, Figure 7c shows a chain mail covering for a sphere, where the changing size of the links is evident. The sphere is cut in half to give a clearer view.

## Conclusion

This short article contains the seeds for many future investigations: How many of the 80 layer group types can be realized in chain mail? Does the answer change when we consider mail with two, three, or more colors? Is it practical to use a 3D printer to print chain mail surfaces? What practical applications might there be for these ideas?

## Acknowledgements

Research for this article was supported by the National Science Foundation under Grant No. DMS-1439786 and the Alfred P. Sloan Foundation Grant No. G-2019-11406 during the Fall 2019 Illustrating Mathematics semester at the Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University. Thanks to Dave Chapman and Edmund Harriss for helping me with *Grasshopper*!

## References

[1] F. Farris. *Creating Symmetry, the Artful Mathematics of Wallpaper Patterns*. Princeton University Press, 2015.

[2] F. Farris. "Wheels on Wheels on Wheels—Surprising Symmetry." *Math. Mag.*, vol. 69, no. 33, 1996, pp. 185–189.

[3] B. Grünbaum and G.C. Shephard. *Tilings and Patterns*. W. H. Freeman and Co., 1987.

[4] G. Hart, "Ball and Chain." (laser-sintered chain mail scuplture). *Bridges Gallery*, Coimbra, Portugal, 27–31 July, 2011. http://gallery.bridgesmathart.org/exhibitions/2011-joint-mathematics-meetings/george-hart.

[5] E. Hitzer and D. Ichikawa. "Representation of Crystallographic Subperiodic Groups by Geometric Algebra." *Electronic Proc. of AGACSE*, 3, Leipzig, Germany, 17-19 Aug. 2008. https://ui.adsabs.harvard.edu/abs/2013arXiv1306.1280H/abstract

[6] A. Jouttijärvi. "The Manufacture of Chain-Mail." I Henriette Lyngström (Ed), *Early Iron*, Netværk for Tidlig Jernteknologi 1, 1996.

[7] Website of M.A.I.L. (Maille Artisans International League). https://www.mailleartisans.org/

[8] T. Verhoeff and K Verhoeff. "From Chain-link Fence to Space-Spanning Mathematical Structures." *Bridges Conference Proceedings*, Coimbra, Portugal, 27–31 July, 2011, pp. 73–80. http://archive.bridgesmathart.org/2011/bridges2011-73.html.