Gauss-Bonnet Sculpting

Edmund Harriss

Department of Mathematical Sciences, University of Arkansas; eharriss@uark.edu

Abstract

The Gauss-Bonnet theorem is a powerful result central to much of modern mathematics. At its heart, it gives the relationship between the curvature of a surface and the amount you turn as you travel around the surface's boundary. This can be used to create a system for sculpture where loops are controlled in order to give the surface form. Starting with paper strips, moving through Curvahedra to a metal sculpture, we explore the creative power of this theorem.

The Gauss-Bonnet Theorem

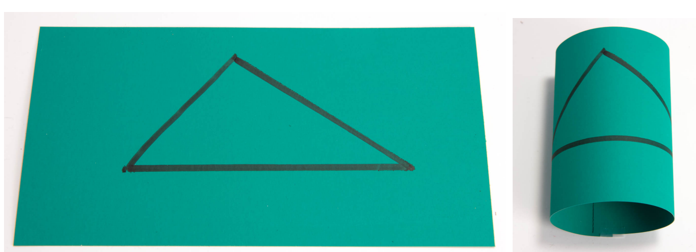
The Gauss-Bonnet (pronounced bo-NAY) Theorem is a generalisation of the classic school geometry result that the sum of the angles of a triangle is 180° [2]. It is one of mathematics' most elegant results about the geometry of the world we live in, but is not broadly known. A statement of the theorem gives a hint as to why, as it requires calculus and a little topology to read.

$$\iint_M K \, \mathrm{d}A + \oint_{\partial M} k_g \, \mathrm{d}s = 2\pi \chi(M)$$

Yet an intuition of the central idea does not require all that mathematical machinery. In particular the symbols above are required to control a very intuitive concept of a surface [8]— any surface, the surface of the earth, the bonnet of a car (the British version of the American "car hood" feels more apropriate here), the skin on your hands, all are surfaces. In fact any three-dimensional object must be surrounded by a surface, though here we will only look at smooth surfaces that do not have sharp corners.

Paper, from Pieces to Strips

A simple example is a flat piece of paper, so let us start there. We also need the idea of paths, and more precisely loops, on this surface. Any line you draw on the paper will do. Generally a closed loop divides the surface up into two pieces, the inside and the outside, just as a surface in general divides three dimensional space into an inside and an outside.



(a) (b)

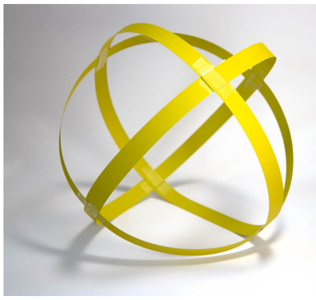
Figure 1: *Triangle on a flat sheet of paper (1a), and the same triangle on rolled paper (1b).*

Note that the word "generally" above is doing a lot of work. There are certainly examples where this is not the case, a rich question to explore in topology. In this case, however, we will restrict ourselves to curves that are separating (they divide the surface into two pieces) and also do not intersect themselves.

On the piece of paper we can draw a triangle (Figure 1a). The angles of this triangle will (famously) add up to 180°. As the paper flexes and bends this remains true; the triangle gets bent alongside the paper, but does not change its shape on the paper, as shown in Figure 1b. These changes are controlled by a closely related theorem that Gauss called the "Theorema Egregium" or honorable theorem [2], that tells how a surface can flex without changing its geometry.

Is it possible to create a triangle with a different angle sum? For drawings on a flat piece of paper, the triangle sum theorem going back beyond Euclid [3] limits this. Gauss' result shows even bending the paper does not help. Yet with strips of paper we can set up a triangle with any length edges and any side lengths. For example, Figure 2a shows a triangle with three equal length edges and 90° angles. Notice that this triangle cannot stay flat, but it could fit onto a sphere. In fact eight such triangles divide up a sphere into equal pieces (Figure 2b).

(a)



(b)

Figure 2: *Equilateral triangle with three* 90° *angles (Figure 2a), and the octahedral sphere made from* 8 *connected together (Figure 2b).*

Angle sum, Turning and Gaussian Curvature

The angle sum of a triangle is closely related to a more general concept, the total turning as you go round a loop. In the case of a flat triangle (on the piece of paper) this must be 360°.

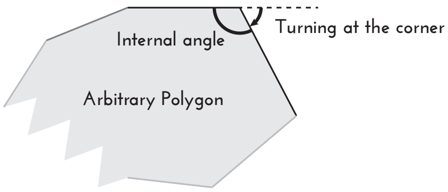


Figure 3: *The internal angle and external angle (turn) at the corner of a polygon.*

The total internal angle of a polygon has a close connection to its turning. At each corner of a polygon the amount needed to turn from travelling along one side to the next is 180° minus the internal angle, as shown in Figure 3. The total turning is therefore 180° times the number of sides minus the total internal angle.

We can now see that this turning for the 90° equilateral triangle (Figure 2a) is 270°. So in a full loop returning to the same position and direction, we have turned less than 360°. Just staying with equilateral triangles (with all side lengths and angles the same) we can make a family of triangles by changing the angle, we see these change from looking like pieces of a sphere, through flat (with a corner of 60°, and so a turning of 360°) to looking like a piece of a saddle (Figure 4).

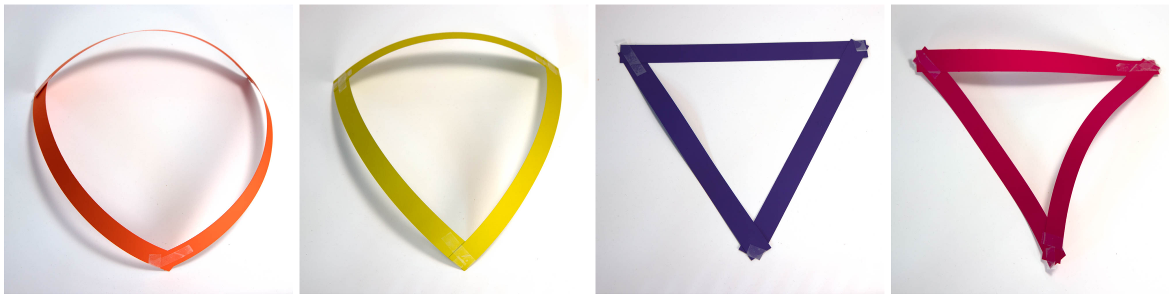


Figure 4: *Equilateral triangles with angles,* 120°, 90°, 60°, 45°, *moving from positive through* 0 *to negative curvature.*

The difference between these three options is described by what is called "Gaussian curvature", with positive curvature (for the sphere), zero curvature (for the flat) and negative curvature (for the saddle). Both positive and negative curvature also have an amount that increases as the radius of the sphere gets smaller or the saddle gets, for want of a better word, saddlier. We can see this in the paper strip models as the angle at the corners increases (to give greater positive curvature) or decreases (to give more negative curvature). Returning to the Gauss-Bonnet formula (Figure 5) we can see that two of its terms are given by the turning round a loop (related to the angles at the corners) and the overall Gaussian curvature.

total turning round boundary

$$\iint_M K \, \mathrm{d}A + \oint_{\partial M} k_g \, \mathrm{d}s = 2\pi\chi(M)$$

total curvature of surface

Figure 5: *Annotated statement of Gauss-Bonnet Theorem*

For the moment we will leave off the final term $\chi(M)$, other than to say it is always a whole number, and for the regions that we are studying so far is always 1. The inclusion of π might hint that the units should be in radians, which we will use for the remainder of this paper. For the regions we have studied therefore the Gauss-Bonnet formula can be summarised as the sum of turning and curvature is equal to 2π. Radians help here to making the clean relationship between curvature and turning.

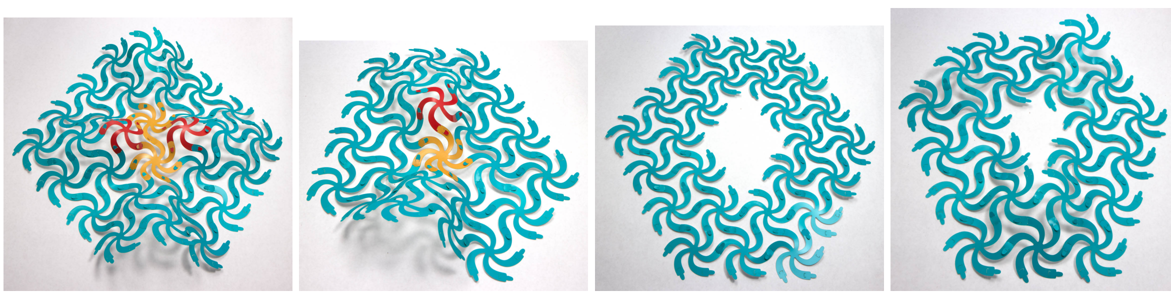
First Sculptures

Figure 6: *Curve with no overall turning.*

As we are now thinking about the turn on the boundary, we do not just have to think about paths with straight edges and turn only at corners; we can instead look at a curved path. For a curved path like Figure 6 the total turn is in fact 0, as we turn one way and then back the other way so we are facing the same direction.

These curves are used in the Curvahedra system that was created by the author, in part to explore the ideas developed in this paper [4]. It was intended as an artistic system that could be used by many people to create attractive objects, while communicating and building intuition of fundamental mathematical ideas; the aesthetics giving a motivation for the mathematical thinking. The curved edges were originally chosen as they seemed to produce more pleasing forms than the more clinical effect of straight strips. They turned out to also introduce a level of flexibility that the straight strips lacked. Their end points can be stretched apart from the flat configuration, which is impossible for a straight strip. Curvahedra pieces will be used for the remainder of this paper, as their precise cut and connection system makes it quicker and easier to create models than stapling or taping paper strips.

Applying the formula we can see that when the $\mathcal{X}(M)$ term is 1 and the turning is 2π, the total curvature must be 0. This does not mean that it is 0 everywhere, as for the triangles above. In fact we can even make a loop that contains 0 curvature, but must have both negatively and positively curved regions within it. As a result the region must contain equal positve and negative curvature regions (Figure 7).



(a) **(b)** **(c)** **(d)**

Figure 7: *Surfaces with total curvature 0.*
(7a), (7b) Two surfaces with equal positive and negative curvature,
(7c) Surface from (7a) with non-zero curvature region removed, lying flat,
(7d) Surface from (7b) with non-zero curvature region removed, but unable to lie flat.

The fact that the curvature can change means that we can use the system to sculpt, creating regions of positive and negative curvature to make any shape that we wish (Figure 8).

(a) **(b)**

Figure 8: *An icosahedral sphere made from 12 5-branch Curvahedra pieces, forming 20 individual curved triangles (8a) and the same pieces coming together in a surface of varying curvature (8b).*

Euler Characteristic

When a surface is closed up something interesting happens. The surface no longer has a boundary, but still clearly has curvature. The sphere in Figure 8a has 20 triangles, each with a boundary with turning of $9\pi/5$. As for the individual triangles, the sum of the curvature and turning is 2π, they have a curvature of $\pi/5$. The total for the 20 triangles is thus 4π. So for this sphere, we have total curvature 4π, and from the formula we have χ equal to 2. Taking a slightly different sphere with four triangles with $2\pi/3$ angles, and thus turning of just π each, we get a curvature of π for each triangle and again a total curvature of 4π. In fact, any sphere that we make will have a total curvature of 4π. Even if we take a distorted sphere, like Figure 8b, with parts of positive curvature and others of negative curvature, then the result has a total curvature of 4π.

This is related to the birth of topology and a result far older than Gauss-Bonnet. The first version is Descartes' theorem [5] that points out that for any polyhedron the total "angle defect" is 4π. The angle defect is the difference of the total angle at a vertex from 2π. If you think about a polyhedron, the individual pieces of the surface are flat, the edges are flat pieces with a fold, so the only place for curvature is at the corners. The angle defect measures that curvature. This result was perhaps regarded as a curiosity until Euler gave a more discrete version. Note that for a flat polygon the total internal angle is $\pi(\text{edges} - 2)$. Knowing the number of edges around each face (and nothing more about its geometry) we can therefore find the total angle available for all the corners. Subtracting that from 2π times the vertices or $2\pi V$ gives the total angle defect.

We can simplify further; we wish to sum $\pi(\text{edges} - 2)$ for every face, but every edge is on exactly 2 faces. So summing the edges on all of the faces will give a total of twice the total edges for the polyhedron or $2E$. Similarly summing -2π over all the faces will give -2π times the total number of faces or $-2\pi F$. The total angle available for the corners is thus, $2\pi E - 2\pi F$ and the total angle defect is $2\pi V - 2\pi E + 2\pi F$. By Descarte's theorem, this is equal to 4π. Dividing by 2π gives the famous Euler characteristic, for any polyhedron that is topologically a sphere:

$$V - E + F = 2$$

In general this value $V - E + F$ is called the Euler characteristic [5] and is an invariant not just for polyhedra, but for any graph drawn onto a surface. The Euler characteristic $\chi(M)$ (for a surface M) is the final term in the Gauss-Bonnet theorem. This draws an amazing link between a geometric property of a surface (its total curvature) and a simple topological property (the Euler characteristic of the surface related to the number of holes passing through it). This idea has proven to be very powerful. Through the nineteenth and twentieth century, its generalisations provided some of the great jewels of differential geometry and topology, such as the the Riemann-Roch theorem [6] and Atiyah-Singer Index theorem [1].

Gauss-Bonnet by Counting

The arguments used above can be stretched further, transforming the calculation of curvature from calculus into simple counting. For example, the triangle surfaces shown in Figure 7, have total curvature 0, established by the turning round the boundary, but this can also be calculated directly from the surface. Notice that the loops are all triangles and the vertices have 5, 6 or 7 branches, but with equal numbers of 5's to 7's.

As the angles at a vertex are all equal, the internal angle at the corner of a 6-branch piece means a turning of $\pi - 2\pi/6 = 2\pi/3$ (120°), so triangles with three such vertices will lie flat. A 5-branch has turning of $\pi - 2\pi/5 = 3\pi/5$, so (assuming all loops have three sides) this introduces five places with too little turn; in fact $2\pi/3 - 3\pi/5$ which is $\pi/15$ too little. Counting all five gives $\pi/3$ too little turning, that must be compensated for with positive curvature of the region. In contrast, a 7-branch has turning of $\pi - 2\pi/7 = 5\pi/7$ so a total of $7(5\pi/7 - 2\pi/3) = \pi/3$. This extra turn cancels out the reduced turn from the 5-branch piece. Note that for this effect to hold all the edges of both the 5 and 7-branch need to be completed into triangles.

The moral here is that in a surface with all triangles a combination of a 5 and a 7-branch will cancel each other out. For an n-branch piece, the angle missing (or extra for negative values) is $n(2\pi/3 - \frac{(n-2)\pi}{n})$ which is $(6-n)\pi/3$. So just counting the branches is enough to work out the curvature; less than 6 branches gives positive curvature, 6 gives zero curvature and more than 6 gives negative curvature. In a surface with only triangles, adding the differences to 6 for all pieces gives the overall curvature. For example one 3-branch will be balanced by three 7-branch, to give a surface with zero curvature.

Increasing the number of edges round a vertex adds extra turning to a loop, decreasing the curvature of the region it surrounds. Similarly adding edges to a loop increases turning. In this case adding an extra corner of a 6-branch to the loop adds turning of $2\pi/3$; twice the effect of adding an extra edge at a vertex. So for a closed surface the curvature can be calculated by counting the difference between 6 and the number of edges at each vertex. Then adding that to 2 times the differences between 3 and the number of edges round each loop. These rules also operate locally, so it is easy to see where a surface has positive or negative curvature.

As an example, consider the torus, which has Euler characteristic of 0, so total curvature of 0. As an aside, the fact that this is the same curvature as the plane is the key to the notion of a covering space, another important tool in modern geometry [8]. The torus in Figure 9a demonstrates this by having twelve 5-branch pieces (on the outside, giving positive curvature) and six loops with four sides (in the centre, giving positive curvature). As adding an edge to a loop has twice the effect of removing an edge from a vertex, the six loops cancel the twelve vertices for a total of zero curvature.

Given that the total curvature of a torus must be zero, it is natural to ask if there is a torus with zero curvature everywhere. The answer is complicated. Such a torus, called the flat torus, definitely exists, but not in 3d. To see this, bend a sheet of paper into a cylinder and then watch it struggle and crumple as you try to bend it into a torus. It is possible to make the Curvahedra variation of the flat torus, as shown in Figure 9b. Every loop of this torus does surround zero curvature, but you can see that it might be hard to fill in the regions with pieces with zero curvature everywhere.

(a) **(b)**

Figure 9: *Two Curvahedra toruses.*

From Theory to Sculpture

To take the theoretical description above and apply it to sculpture we simply need a material that will behave like the paper and mylar. A sheet material that is quite rigid as a surface, but will bend easily at right angles to it. Though sticking with Mylar already allows relatively large objects (Figure 10). Once the material is chosen, it can be used to make loops, which are locally flat. The total turn around a loop will control the curvature and placing loops together builds up a surface. For many materials the physical properties of the material will bend the loops into pleasing shapes without additional work. For example thin sheet plastic.

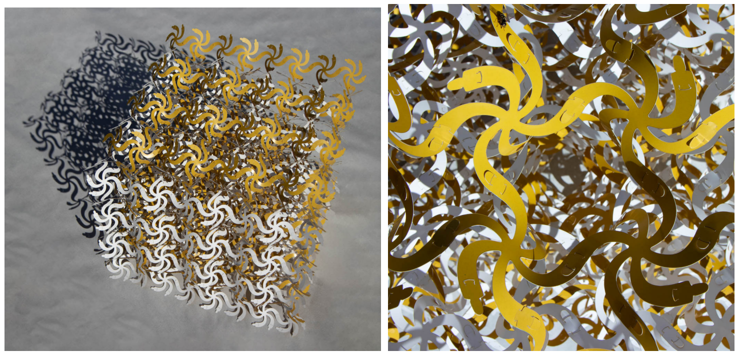


Figure 10: *2-foot cube approximating a Schwartz P-surface [7], constructed with the Museum of Mathematics and help from conference goers at JMM 2019 and NCTM 2019.*

A great challenge for larger scale is sheet metal. Although there are springier varieties that might work directly as above, mild steel introduces two problems. It can distort, bending in plane, thus changing the geometry. More seriously it will bend into shape and will prefer to do so at a place where it is already bending. Thus without tools it is hard to get smooth bends over a long piece of the steel. Techniques such as roll bending have been developed to create smooth bends [10]. Note that wheeling using an English wheel can take advantage of the ability of the metal to stretch in its surface, so is actually able to change the local geometry of the steel to give compound curvature (generally positive). Hammering can have a similar effect, and even produce beautiful negative curvature, for example in the work of Benjamin Storch (Figure 11) [9].

Figure 11: *Negative curvature in a stainless steel Möbius Strip, created by Benjamin Storch. Photo ©Benjamin Storch, used with permission. http://www.benjaminstorch.com/?project=spiral-moebius*

The key to all these techniques is the gentle bending of the metal that must happen over an area. Using the Gauss-Bonnet theorem, however it is possible to smoothly exert force over a loop while interacting at a small number of points. The method is to create a loop from several pieces, but with loose connections so no piece has to bend, for example with long bolts, then slowly tightening at each connection point. As the loop gets gently tightened the main force travels in the plane of the surface which does not want to bend, and so is smoothly distributed over the loop. As a result smooth bends can be achieved without tools beyond the wrench to tighten the bolts (Figure 12).

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1439786 and the Alfred P. Sloan Foundation award G-2019-11406 while the author was in residence at

(a) **(b)**

Figure 12: *A metal curvahedron ball in the collection of the Univeristy of Arkansas Honors College (12b), and the Ball in construction (12a).*

the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Illustrating Mathematics program.

The metal sculpture shown in he final section was created as a comission for the University of Arkansas Honors college, from a gift from Dean Lynda Coon. Both prototypes and the final pieces were made by Emily Baker of the Fay Jones School of Architecture at the University of Arkansas.

References

[1] M.F. Atiyah and I.M. Singer. “The index of elliptic operators. I”. *Ann. of Math.* Series 2, Vol. 87, No. 3, 1968, pp. 484–530. Available at https://doi.org/10.2307/1970715.

[2] T. Banchoff and S. Lovett. *Differential geometry of curves and surfaces*. A.K. Peters, 2010.

[3] O. Byrne and B. Rogers. *The first six books of the elements of Euclid, in which coloured diagrams and symbols are used instead of letters for the greater ease of learners*. William Pickering, 1847.

[4] E.O. Harriss. *Curvahedra*, 2018. Available at https://curvahedra.com.

[5] D.S. Richeson. *Euler’s gem: the polyhedron formula and the birth of topology*. Princeton University Press, 2008. Available at http://www.loc.gov/catdir/enhancements/fy0834/2008062108-b.html.

[6] B. Riemann. “Theorie der Abel’schen Functionen”. *J. Reine Angew. Math.* Vol. 54, 1857, pp. 115–155. Available at https://doi.org/10.1515/crll.1857.54.115.

[7] A. Schoen. *Infinite Periodic Minimal Surfaces Without Self-intersections*, NASA technical note, National Aeronautics and Space Administration, 1970. Available at https://books.google.com/books?id=LxOb_NlMLtMC.

[8] R. Schwartz. *Mostly Surfaces*, Student mathematical library, American Mathematical Society, 2011, Available at https://books.google.com/books?id=sS2IAwAAQBAJ.

[9] B. Storch. “Anticlastic Form - Manifesting Fields of Tension.” *Bridges Conference Proceedings*, Donostia, Spain, 2007, pp. 189–194. Available at http://archive.bridgesmathart.org/2007/bridges2007-189.html.

[10] R.H. Todd, D.K. Allen, and L. Alting. *Manufacturing processes reference guide*. Industrial Press, 1994. Available at http://www.loc.gov/catdir/enhancements/fy1511/93031767-d.html.