The Hairy Klein Bottle

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Abstract

In this collaborative work between a mathematician and designer we imagine a hairy Klein bottle, a fusion that explores a continuous non-vanishing vector field on a non-orientable surface. We describe the modeling process of creating a tangible, tactile sculpture designed to intrigue and invite simple understanding.

Figure 1: *The hairy Klein bottle which is described in this manuscript*

Introduction

Topology is a field of mathematics that is not so interested in the exact shape of objects involved but rather in the way they are put together (see [4]). A well-known example is that, from a topological perspective a doughnut and a coffee cup are the same as both contain a single hole. Topology deals with many concepts that can be tricky to grasp without being able to see and touch a three dimensional model, and in some cases the concepts consists of more than three dimensions. Some of the most interesting objects in topology are non-orientable surfaces. Séquin [6] and Frazier & Schattschneider [1] describe an application of the Möbius strip, a non-orientable surface. Séquin [5] describes the Klein bottle, another non-orientable object that "lives" in four dimensions from a mathematical point of view.

This particular manuscript was inspired by the hairy ball theorem which has the remarkable implication in (algebraic) topology that at any moment, there is a point on the earth's surface where no wind is blowing. We will introduce our object, "The hairy Klein bottle", which we believe can give a "face" to mathematical ideas, reveals this beautiful concept to a wide and varied audience with no mathematical background.

The Hairy Klein Bottle as a Tool

Our object will be to explain the following definitions in a simple manner.

Definition 1. A vector field $\vec{F}$ is a function continuously mapping a point P on a surface S to the respective tangent vector at P, locally we denote it by $(P, \vec{F}(P))$.

Definition 2. An insulated zero is a point that maps to the zero vector and whose neighborhood is mapped to non zero vectors.

Think of the vector field as a hairy surface as seen in Figure (1). At every point on the surface there exists a respective hair which is the tangent vector. The hairs (the vectors that form the vector field) can be oriented in different directions on the given surface. If the hairs have uniform length (length 1), we will say that the vector field is a unit vector field. Combing the hair in different directions can lead to a tuft or "cow lick" areas which yield a vector field that is non continuous.

Orientability is a property of surfaces that measures whether it is possible to make a consistent choice of surface vector field at every point. For example a sphere is orientable, since we can assign one color to the outward facing surface of the sphere, and another color for the inner surface. All orientable surfaces can be painted using two colors.

We can conclude that a non-orientable surface is a surface with only one side. The most famous example of a non-orientable surface is the Möbius strip, which is a surface with only one side and only one boundary. To construct a Möbius strip, take a finite strip (a paper strip), twist it one-half a turn and glue the strip end to end. This construction allows us to travel along the surface in a continuous way and still reach every point on the surface. In the Möbius strip there exists a point such that the direction is changed from clockwise to counter-clockwise. Gauld [2] and Séquin [5] provided a mathematical background and design application for the Möbius strip.

Continuous deformations are functions of stretching, crumpling and bending (which can be defined as a function). Tearing or gluing do not create continuous deformations. We will say that a mug and a doughnut are topologically identical, since there exists a continuous mapping (which is a deformation) that can turn one shape into the other. The shape's topology is preserved - in this case a shape with a single hole.

A Klein bottle is an extension of a Möbius strip, which was first described in 1882 by the German mathematician Felix Klein. It is constructed by joining the edges of two Möbius strips together. The Klein bottle is also non orientable, but unlike the Möbius strip it does not have an end, i.e., there are no boundary points. In Figure (2) the process to obtain a Klein bottle is described (for more details see [2]). It is a

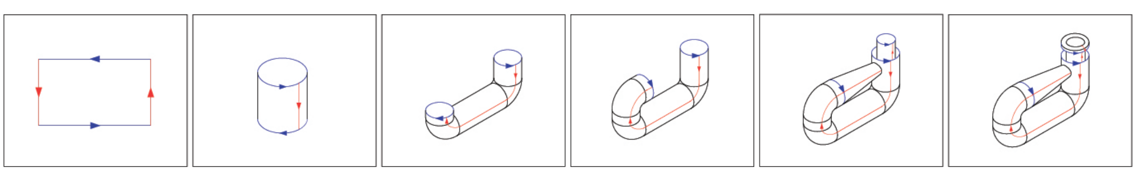


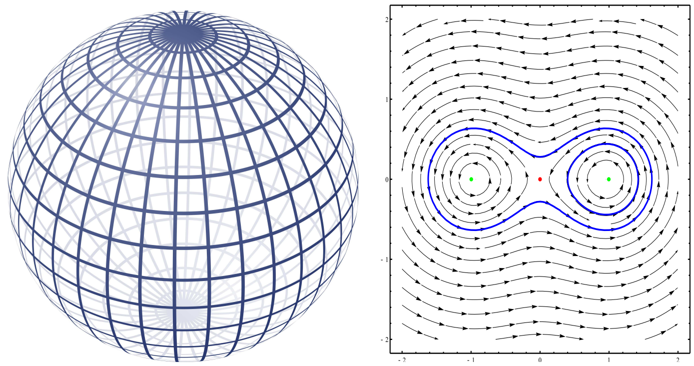
Figure 2: *Constructing the Klein bottle*

mathematical object, not a physical object, i.e., it has properties that cannot be realized. Since the Klein bottle intersects itself, the intersection points on the surface are obtained twice (the second time is when we punch the hole). To solve this problem (which is legal from a mathematical perspective) we add another dimension - a fourth dimension, that preserves the object's continunity.

The beautiful Poincaré-Hopf theorem relates the properties of vector fields to the topological structure.

Theorem 3. *Let S be an orientable smooth surface and $\vec{F}$ a vector field on S with isolated zeroes, then*

$\sum_i Index_{x_i} \vec{F} = \chi(S)$, where the sum of the indices is over all the isolated zeroes of $\vec{F}$ and $\chi(S)$ is the Euler characteristic of S (see Figure(3a)).



(a) **(b)**

Figure 3: *(a) Given a map on a manifold, F is the number of faces (polygons), V is the number of vertices and E is the number of edges, the Euler characteristic is defined as $\chi(S) = V - E + F$ (Figure from Geek3 (Wikipedia)).*
(b) The colored point represents zeroes of the vector field, the index of every point is the number of counterclockwise rotations the vector field makes around the point (and is negative for clockwise, as for the red point) (Figure from Radost Waszkiewicz (Wikipedia)).

From the Poincaré-Hopf theorem it follows that only a surface of characteristic zero can host a vector field which is nowhere zero. For example the sphere has $\chi = 2$ and thus can not host a nowhere zero vector field (the hairy ball theorem), whereas the torus has $\chi = 0$ and can host a vector field. The Klein bottle also has $\chi = 0$ and due to that maintains a continuous vector field. This can be easily understood in our model simply by combing the hairs.

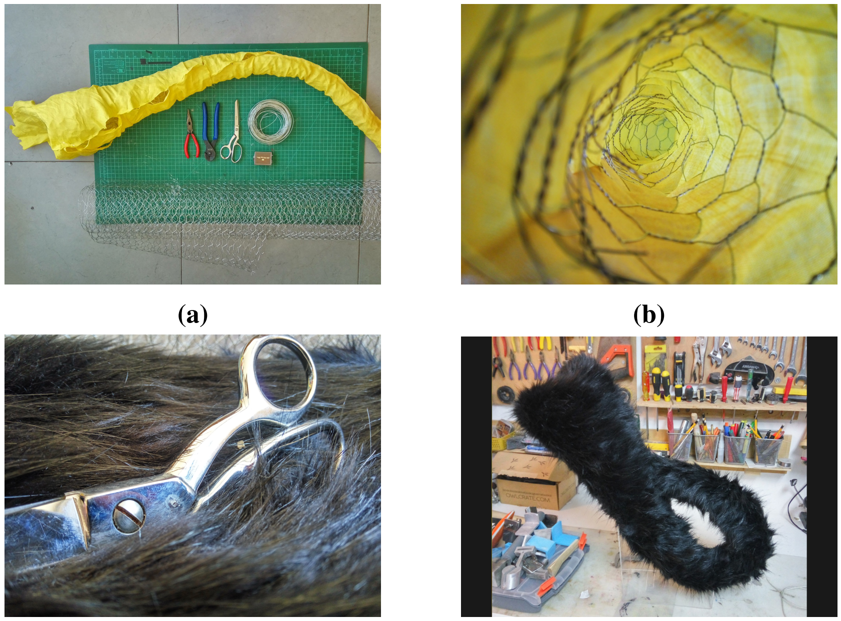
Design Process and Realization of Mathematics by Prototype

The sculpting of the Klein bottle began by creating a frame in the desired shape using a wire mesh that would serve as the base form. The ends of the bottle were sewn together with strips of 1mm pliable steel wire. The steel frame uses a hexagonal lattice, which has been proven the optimal polygonal lattice in the sense of strength. This is due to the fact that the hexagonal regular lattice gives maximum area and minimal perimeter (comparied to, for example a square lattice). We thought it essential that the object would be relatively robust since it was essential for the model to be percieved as playful, and inviting.

This base form was then covered in a stiff cotton fabric. The layer of cotton was intended to isolate the shaggy textile from the sharp wire structure thus allowing it to be pulled over the base form like a sleeve.

Finding the right fabric was essential. We must have combed every fabric in the shop (much to the confusion of the saleswoman) looking for the perfect representative of vectors on the surface. After selecting the fabric, it was measured out and cut to form. The next step was to sew the layer of hair. It was sewn first as a sleeve and then slid over the thick cotton fabric. Next, at the mouth of the bottle the fabric is folded over and sewn together with the intersecting end. This allowed for a continuous flow of fabric. This detail was important because it allowed the more curious to reach inside and easily comprehend the continuity of the surface. It was crucial that the bottle would be more that simply an explanatory sculpture. It needed to act as an interactive tool that it would invite you to touch it, stick your hand through and of course try to comb it.

For the final touch we bent a sheet of 4mm acrylic to serve as a base to keep the bottle in place. Now, due to this collabortive effort, one can see it is possible to comb the hair on the Klein bottle in same direction



(a) **(b)** **(c)** **(d)**

Figure 4: *Our process: (a) The desired frame, (b) make it rigid by a deformation of hexagonal lattice, (c) The hair, which represents the tangent vector field, (d) the desired result.*

such that no bald spots or "cowlicks" appear. This leads to the understanding that indeed a non-vanishing vector field on a Klein bottle exists.

The hairy Klein bottle was presented at the Symmetry and Fractals exhibition at the Holon Institute of Technology alongside a short explanation. It was impressive how easily such a complex subject could be grasped by the visitors to the exhibition, some of whom had no mathematical background, simply by observing and interacting with the model. The Design and Mathematics departments are enjoying some very successful collaborations and intend to keep developing communication methods that provoke and intrigue those with a lesser mathematical background and engage a wider audience allowing the wonders of the mathematical world to be revealed.

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