Interwoven Islamic Geometric Patterns

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Abstract

Rinus Roelofs has exhibited numerous planar and polyhedral sculptures made up of two layers that weave over and under each other to form a single connected surface. I present a technique for creating sculptures in this style that are inspired by the geometric structure of Islamic star patterns. I first present a general approach for constructing interwoven two-layer sculptures, and then specialize it to Islamic patterns in the plane and on polyhedra. Finally, I describe a projection operation that bulges the elements of these designs into undulating dome shapes.

1 Introduction

For more than a decade, mathematical artist Rinus Roelofs has contributed a sequence of exciting geometric ideas to the Bridges conference. His sculptures, whether physical objects or computer renderings, always deliver novel aesthetic twists based on natural mathematical starting points.

I am particularly fascinated by a theme that Roelofs introduced in his 2008 paper "Connected Holes" [4], and revisited in different forms in subsequent papers [5, 6]. I refer to these constructions as "interwoven surfaces", after his use of the term in his 2008 paper. Interwoven surfaces share a few common features. They are based on a tiling of the plane or a polyhedron (i.e., a tiling of the sphere). Each tile is converted into a small module, consisting of two layers of geometry with arms that reach to the edges of the tile. When the tiles are joined, the arms of the modules fuse, with each "upper layer" connecting exclusively to the "lower layer" of its neighbouring tiles, and vice versa. The resulting 3D forms are topologically fascinating. The boundaries of the holes that remain at the vertices of the tiling are knots or links, and often the many upper and lower layers merge to become a single connected object. Mathematically, these sculptures might be interpreted as physical models of branched coverings, not unlike those explored by Schleimer and Segerman in the context of spherical photography [7].

In 2016 I was given the opportunity to display a selection of my 3D printed spherical Islamic star patterns, based on earlier research [3]. I wanted to supplement my existing collection with a large new centrepiece, ideally exploring mathematical techniques I had not tried before. At Bridges that year, Roelofs not only showed a few interwoven surfaces, he distributed laser-cut paper templates that could be folded into an interwoven cube and tetrahedron (Figure 1). I resolved to explore whether Islamic geometric patterns could be realized as interwoven surfaces. These paper templates provided perfect models for contemplation as I considered the problem.

This paper offers a construction technique for Roelofs-style interwoven surfaces, and adapts that technique for the special case of Islamic geometric patterns. I build up from surfaces based on planar tilings, to polyhedra, to more rounded forms based on a geometric "bulging" operation. I show a combination of physical and virtual models constructed using these techniques.

2 Interwoven Planar Designs

In this section I demonstrate a construction that yields an interwoven surface from a patch of regular polygons connected edge-to-edge, together with suitably chosen motifs inscribed in every polygon. This construction is

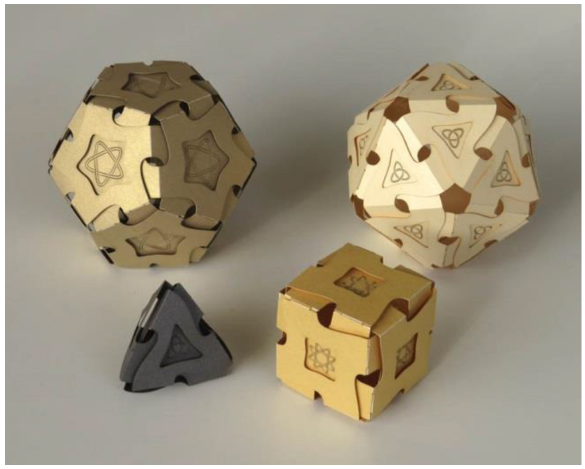

Figure 1 : *Four two-layer paper sculptures by Rinus Roelofs, based on four of the Platonic solids. The octahedron is not compatible with this construction.*

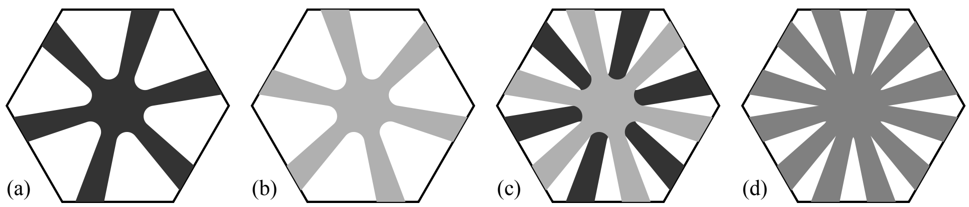

Figure 2 : *A demonstration of a motif that can be used to construct an interwoven surface. The original motif M is shown inscribed in a hexagon (a), followed by its reflection M' in a line through the centre of the hexagon (b), the two copies superimposed (c), and a version with colour distinctions erased to demonstrate d_n symmetry (d).*

not fully general (for example, Roelofs' original paper includes examples based on non-regular tiles), but it is rich enough to create a wide variety of interesting surfaces.

Let us restrict our attention to a single n-sided regular polygon P, with the aim of constructing a 3D module that will mate naturally with those of neighbouring polygons when they are brought together.

For our purposes, we define a *motif* M to be a 2D shape contained entirely within P, satisfying the following conditions:

1. M has c_n symmetry. That is, it has n-fold rotational symmetry about the centre of the polygon, but no reflection symmetries.
2. M is a connected set. M may have internal holes—it need not be simply connected.
3. The intersection of M with each edge of P consists of a positive, finite number of disjoint line segments. We refer to the collection of all of these line segments as the "arms" of M. The rotational symmetry of M implies that every edge of P will contain a congruent set of arms.
4. Let M' be the motif obtained by reflecting M across any of the lines of reflection symmetry of P. Note that if M satisfies the three conditions above, so will M'. We require one final constraint: that the arms of M' be disjoint from the arms of M.

Figure 2 gives a simple example of a 6-fold motif that satisfies these conditions. Note that the construction is based on simple closed curves, which lead to the arms on the polygon's boundaries. In some cases we perform the construction using only open lines or curves, and thicken the resulting 3D paths into tubes afterwards.

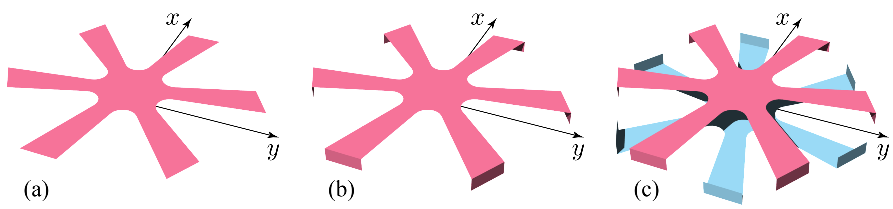

Figure 3 : *The construction of a module from a motif. The original motif is raised above the plane (a), rectangles are added to form the upper layer (b), and a copy is placed as the lower layer (c).*

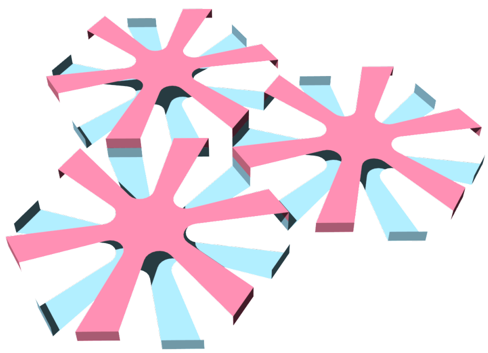

Figure 4 : *Three linked copies of the module of Figure 3. Note that the curve formed by the linked boundaries of the modules around the shared vertex has the topology of a trefoil knot.*

Now consider a motif M, drawn in the x-y plane in 3D space, with the centre of the motif at the origin. Lift M by some small distance in the z direction, so that it lies above the plane. For every arm of M, i.e., every line segment where M meets the boundary of P, we construct a vertical rectangle by sliding that line segment back down to the x-y plane. I refer to this this translated copy of M as the "upper layer" in a module. To it, add a "lower layer" by taking a copy of the upper layer rotated by 180 degrees about the x axis. The resulting module will consist of two disjoint pieces provided M satisfies the conditions above. Furthermore, when viewed from directly above, the projection of the module onto the x-y plane will have d_n symmetry (all the rotation and reflection symmetries of P). Figure 3 illustrates the steps in this construction process.

When two copies of P are placed edge-to-edge, their modules will naturally link up, with the upper layer of one module connecting to the lower layer of its neighbour and vice versa. These connections arise because the rectangles defined above create a pattern of line segments with d_n symmetry where they meet the x-y plane.

This construction can be extended to any number of regular polygons joined edge-to-edge in a patch, provided the polygons all use compatible motifs, i.e., motifs with equivalent patterns of arms. The result will be a 3D model of an interwoven sculpture made from an idealized material with no thickness, as in Figure 4. The added vertical rectangles also create unattractive creases in the design. In Section 5 I will discuss how to "bulge" the layers continuously, so that modules can be thickened and joined into a smooth sculpture.

Although this construction always yields an interwoven model in Roelofs' style, the model is not necessarily connected. Let us assume that the patch of polygons used above as scaffolding is simply connected, and that it contains at least one "internal vertex" (a vertex not on the boundary of the patch). If the polygons in this patch can be coloured black and white so that polygons sharing an edge have opposite colours (or equivalently, if all internal vertices are surrounded by an even number of polygons), then the structure naturally falls apart into two components. One component consists of white upper layers and black lower layers, and the other of black upper layers and white lower layers. If the patch contains even a single internal vertex of odd degree, then this two-colour decomposition is impossible and the model will consist of a single connected surface.

3 Interwoven Star Patterns

The construction in the previous section can be specialized to produce designs in the style of Islamic geometric patterns, simply by choosing suitable motifs for the polygons. A natural starting point is to inscribe a regular n-gon with a motif built around a central, rotated n-pointed star. When the star is rotated at the correct angle and combined with its reflection, the two copies will conspire to produce a single $2n$-pointed star under projection. If the unrotated star has points that are aligned with the midpoints of the edges of an n-gon, then it must be rotated by $90°/n$ to complete this construction. In this way we can imagine a novel decomposition of a star pattern not into interlaced bands, as we would expect in Islamic art, but into out-of-phase layers.

An example is shown in Figure 5. We start with a regular hexagon, containing a six-pointed star as the beginning of a motif. The star is rotated by $15°$, and the edges meeting at its outer points are extended until they meet the edges of the hexagon. This motif combines with its mirror image to produce a twelve-pointed star. Copies of this module can combine to create a fairly natural Islamic star pattern that is also expressible as an interwoven surface.

Scanning through the historical record, we find that the resulting star pattern is nearly identical to that found in Plate 93 of Bourgoin's collection [1]. In Bourgoin's drawing, the 12-pointed stars are surrounded by regular dodecagons, further subdividing the pattern and creating small four-pointed stars as a by-product. We can easily modify the construction shown here, adding the ring to the motif, as shown in Figure 6. In that example, I create variety by adding the ring to only one of the two layers.

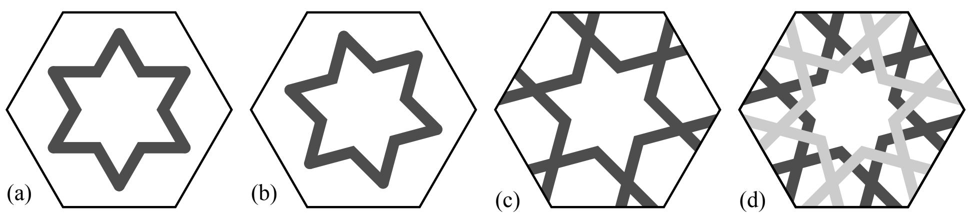

Figure 5 : *The construction of a motif suitable for interwoven star patterns. A star is drawn inside a hexagon, with its points facing the midpoints of the hexagon edges (a). The star is rotated by* $15°$ *(b), and the edges at its outer points are extended (c). When combined with its mirror reflection (d), the resulting drawing includes a 12-pointed star.*

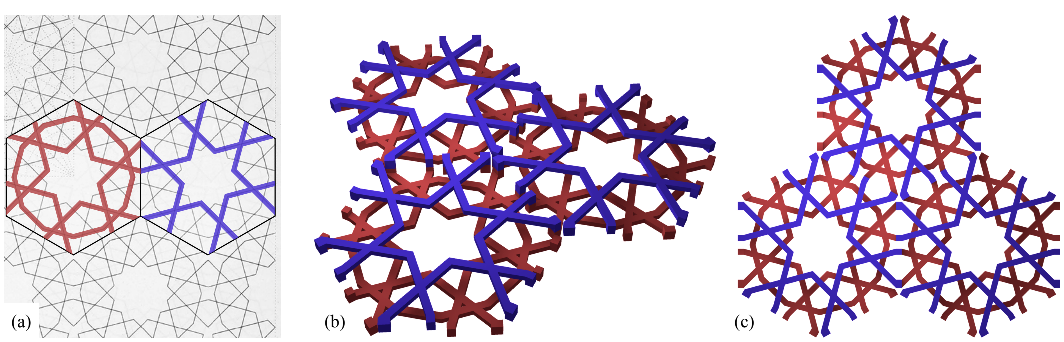

Figure 6 : *An interwoven star pattern based on Bourgoin's Plate 93. Bourgoin's original drawing is shown in (a). Two hexagon regions are highlighted; the red and blue designs are used as motifs for the lower and upper layers, respectively. Three copies of the resulting module are shown in perspective in (b) and from directly above in (c). The motifs are thickened into rectangular tubes, and small rectangular prisms are added at the arms to allow modules to meet.*

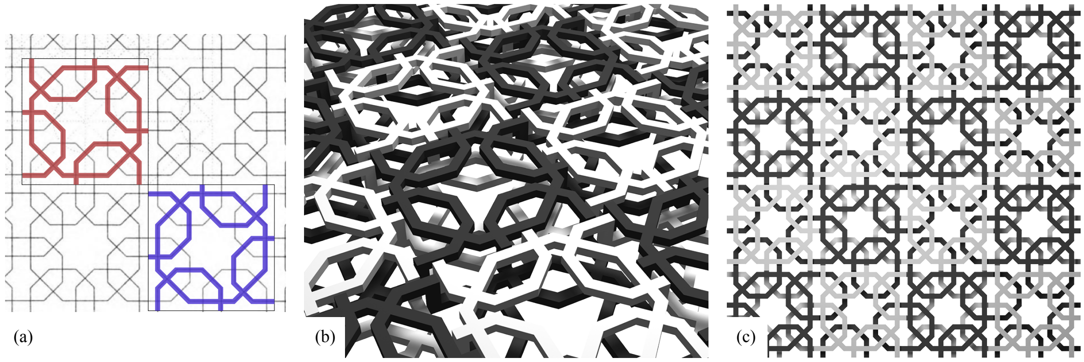

Figure 7: *A planar interwoven sculpture based on Bourgoin's Plate 67. Bourgoin's drawing is shown in (a), with a copy of the motif and its reflection highlighted. The interwoven model is shown in perspective in (b) and from above in (c). The motif copies are coloured to highlight that this sculpure is made from two disconnected components.*

We can also work in the opposite direction, and search for historical patterns that lend themselves to a two-layer decomposition. Figure 7 shows an example based on Bourgoin's Plate 67. This pattern can be seen as composed from a grid arrangement of regular octagons linked by smaller, irregular octagons. A square tile, placed as shown, surrounds a single regular octagon and divides eight irregular octagons in half. These eight half-octagons can be broken into two families that are equivalent under a reflection. Either of those families can be combined with the regular octagon to produce a motif. In this case, both the motif and its reflection need to have a copy of the regular octagon, in order to ensure that the design remains connected.

The bands in these designs are not interlaced in the traditional sense found in Islamic art. Normally, when two paths cross in a pattern we designate one to pass over the other, and assign crossings so that a path crosses alternately over and under the bands it intersects. Here, the weaving occurs at a coarser level, producing a hybrid of traditional interlacement and Roelofs' interwoven surfaces. I find the result to be aesthetically interesting and worthy of further experimentation as part of the visual language of Islamic art. Furthermore, this style of weaving solves a practical problem. Much as it would be natural to convert the 2D depiction of interlacement into 3D ribbons that weave over and under each other, the resulting model would consist of a large number of disconnected bands, and a 3D print of the model would fall apart. With interwoven surfaces it is easy to produce a sculpture comprising a single connected component, making these sculptures a natural fit for 3D printing.

4 Polyhedral Sculptures

The technique of Section 2 uses a patch of regular polygons as a scaffolding, leading to a surface in the plane (or rather, in a thin slab surrounding a plane). A natural extension of that technique is to fold the regular polygons in a patch into a polyhedron, and link neighbouring modules to form an interwoven two-layer polyhedral surface. In this case we might more comfortably refer to the layers as "inner" and "outer" rather than "upper" and "lower".

The square motif of Section 3 lends itself naturally to use on the faces of a cube, as shown in Figure 8. In this case we must extend the arms of the inner layer outward until they encounter the arms from the outer layers of neighbouring cube faces. Unlike the 3D rendering in Figure 8(b), in the 3D printed version I terminate these extended arms in small quarter-cylinder "elbows". Note that whereas the planar version of this design will lead to two disconnected components, the cubical version is a single knotted surface.

Figure 9 shows another polyhedral design. I begin with a pentagonal motif, constructed in the same manner as the hexagonal motif in Figure 5. When short extensions are added to the arms of the motif, copies of the resulting module can assemble into a dodecahedral configuration.

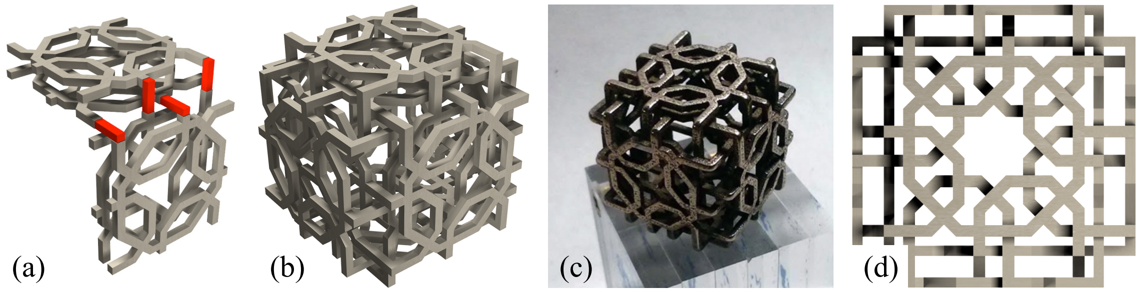

Figure 8 : *A cubical interpretation of the design shown in Figure 7. Two modules are placed with a 90 degree fold between them in (a). Extra struts (shown in red) connect the modules. The finished cubical model is shown rendered in (b), and 3D printed in bronze in (c). A top-down projection is shown in (d), recovering the original pattern.*

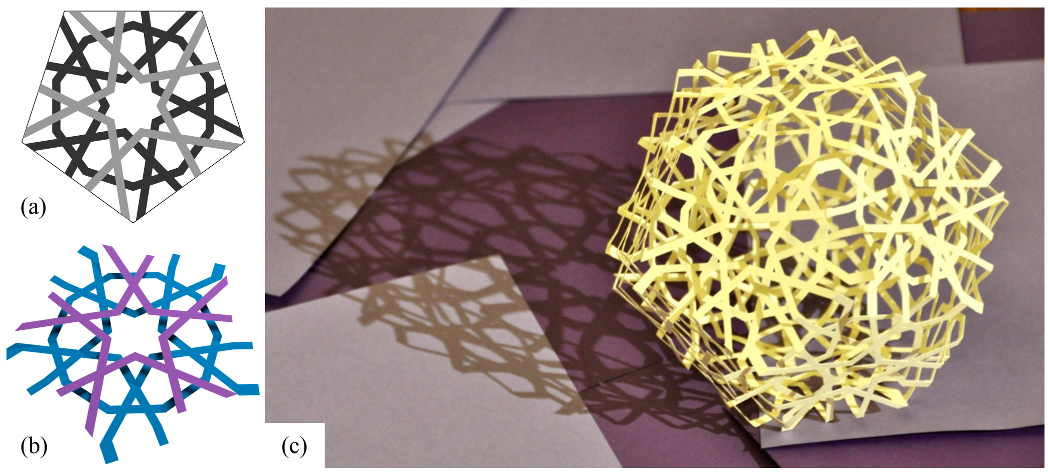

Figure 9 : *A dodecahedral sculpture constructed from twelve copies of a pentagonal module. The 2D motifs are shown in (a), a 3D rendering in (b) with small extensions added to the lower arms, and a photograph of a paper sculpture in (c).*

5 Domed Sculptures

The proof-of-concept designs shown in the previous section all included creases where extra geometry was added to join neighbouring modules. In the polyhedral case, these creases are a reasonable part of the aesthetic, but in the plane they are a distraction, and my original hope was to create smoother spherical sculptures to match my previous research.

I have experimented with a solution based on projecting a motif orthogonally onto a "dome", creating a bulged version of the motif carved out of the dome's surface. To do so I require a means of constructing a smooth "n-dome", a function specified as a height field. The n-dome should reach its maximum at the centre of a regular n-gon and curve downward to meet the plane at the n-gon's boundary. I take inspiration from a "pillow" function defined by Bourke [2], which satisfies my requirements for the special case of a square (Figure 10, left). His function is defined parametrically, but can be re-interpreted as the product of height fields corresponding to two cylinders with orthogonal axes. This construction generalizes naturally to n-gons. I define a height field corresponding to a quarter cylinder joined at its top to a horizontal plane (Figure 10, right). The product of n such fields, rotated to lie on the edges of a regular n-gon, yields domes like those in Figure 11. By projecting motifs onto these domes we can create interwoven planar patterns like the one in Figure 12(a).

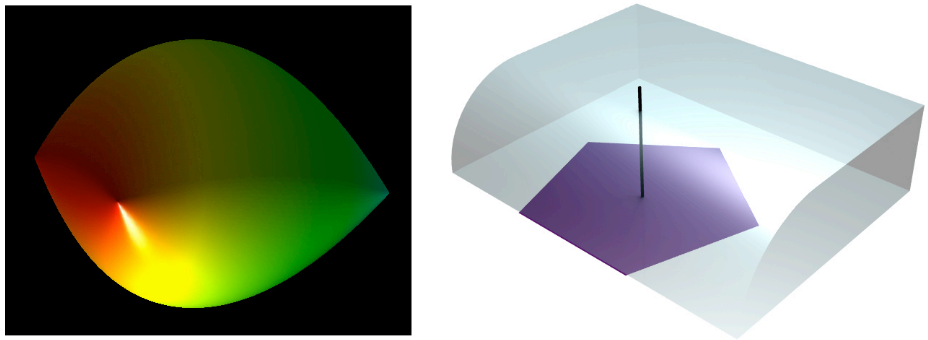

Figure 10 : *Bourke's rendering of the pillow function is shown on the left. On the right, a height field is constructed from a quarter cylinder and a plane, aligned with a single edge of a regular polygon. The product of five such height functions yields the pentagonal dome shown in Figure 11.*

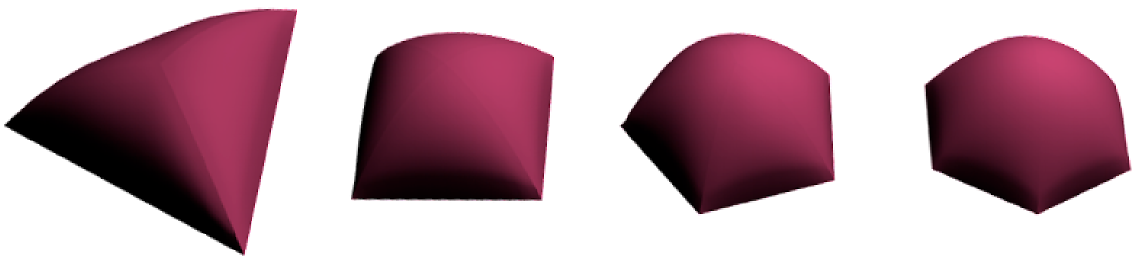

Figure 11 : *Examples of smooth domes with different orders of rotational symmetry from three to six.*

The dome function defined above can be expressed as the product of n simpler functions, each one defined in terms of the distance from a point to an edge of the polygon. We can perform a similar bulging operation on a polyhedral module to create a smooth spherical sculpture. In this case, we simply measure distance on the surface of a reference sphere rather than in the plane, and project points radially from the sphere's centre. Figure 12(c) shows an example based on a truncated icosahedron, with smooth spherical versions of the hexagonal and pentagonal modules shown in Figures 6 and 9.

In both cases, we turn the dome projection into a proper 3D model by extruding it vertically (or radially) by a desired thickness. This operation is somewhat ill-founded mathematically, as the implied thickness of the extrusion will depend on the slope of the dome at every point. A more rigorous version of this operation would extrude by an amount that varies with the local orientation of the normal of the dome surface. However, as long as the dome never becomes too steep, this simple approximation is sufficient.

6 Conclusion

I have presented a collection of techniques that bring together constructions of Islamic geometric patterns with the interwoven surfaces introduced by Rinus Roelofs. There are many ways to explore this space further, uncovering other families of Islamic patterns that lend themselves to this treatment. It would also be worth exploring extensions to more than two interwoven layers.

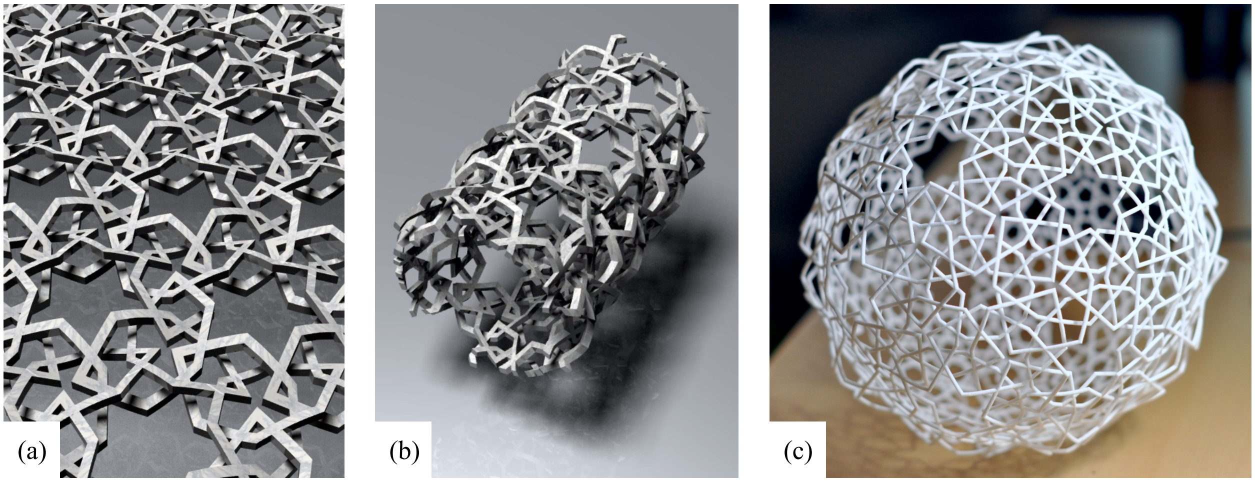

Figure 12 : *(a) A bulged interwoven pattern in the plane. (b) The same pattern wrapped into a cylinder. (c) A 3D printed spherical interwoven star pattern based on a truncated icosahedron.*

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