On Infinite Kepler-Poinsot Polyhedra

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Abstract

Polyhedra are a standard math-art topic, but the Kepler-Poinsot solids and the infinite Petrie-Coxeter polyhedra are less emphasized. Their combination is even entirely new, and so it happened a new regular polyhedron, of infinite Petrie-Coxeter and Kepler-Poinsot type, was recently discovered. The present paper explores this case and two more: a tetrahedral, octahedral and an icosahedral symmetry case. It provides an example of an infinite Kepler-Poinsot solid in each case. It discusses their construction and, if possible, their generalized Euler-Cayley-formula.

Introduction

Cayley generalized the formula of Euler (and Descartes) for a convex polyhedron with V vertices, F faces and E edges to star polyhedra, using the face density a, the vertex figure density b, and the polyhedron density c (see [3] or [8] for definitions of these notions):

$$bV + aF - E = 2c.$$

For instance, for the small stellated dodecahedron, the {5/2, 5}, $V = 12$, $F = 12$, $E = 30$, $a = 2$, $b = 1$, $c = 3$, so that $1\times12 + 2\times12 - 30 = 2\times3$, while the numbers for V and F switch for its dual, the {5/2, 5}. For regular convex polyhedra all density values equal 1, and thus Cayley's formula is a generalization of Euler's formula. For higher genus polyhedra there is another generalization of Euler's formula:

$$V + F - E = 2(1 - g).$$

Here, g stands for the genus (see [2] for a definition). For instance, for the Petrie-Coxeter {6, 4}, $V = 12$, $F = 8$, $E = 24$, and $g = 3$ so that $12 + 8 - 24 = 2\times(1 - 3) = -4$ while the numbers again switch for the dual, the {4, 6}.

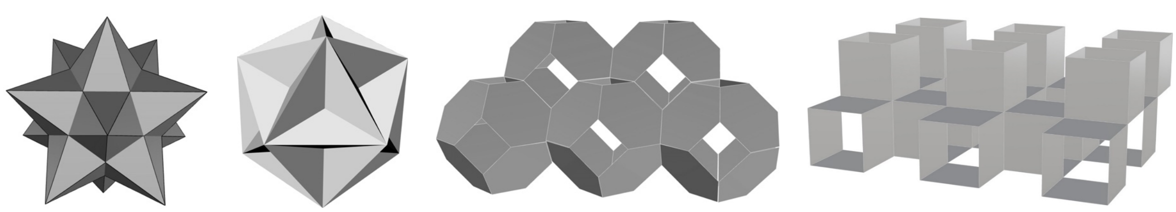

Figure 1: *Two Kepler-Poinsot polyhedra, the {5/2, 5} and the dual {5, 5/2}, and two Petrie-Coxeter polyhedra, the {6, 4} and the dual {4, 6}.*

It was suggested (see [4]) that the two extensions of Euler's formula, that is, Cayley's formula and the higher genus expression, are to be combined a single formula of the type

$$aV + bF - E = 2c(1 - g).$$

Before even trying to establish this formula, it is good to look at some examples, of course (see [7]). Here we examine three new polyhedra of infinite Kepler-Poinsot type and show this generalized Euler-Cayley formula can be applied in all but the last case. Yet, more interesting than this number work are perhaps the representations of these new polyhedra, in particular for the artistically inclined reader, as they have an intrinsic beauty. We give an example for each symmetry case, the tetrahedral, octahedral and icosahedral symmetry. All polyhedra presented here are new, except the octahedral case (see [6]). We begin with this case, since it summarizes some earlier results.

An Example with Octahedral Symmetry

A new regular (compound) polyhedron was obtained using adjacent open cubohemioctahedra, that is, series of cubohemioctahedra of which the squares are removed (see [6]). They are joined on these removed squares, so that eight hexagons meet at each vertex: its Schläfli symbol is {6, 8} (see Fig. 2).

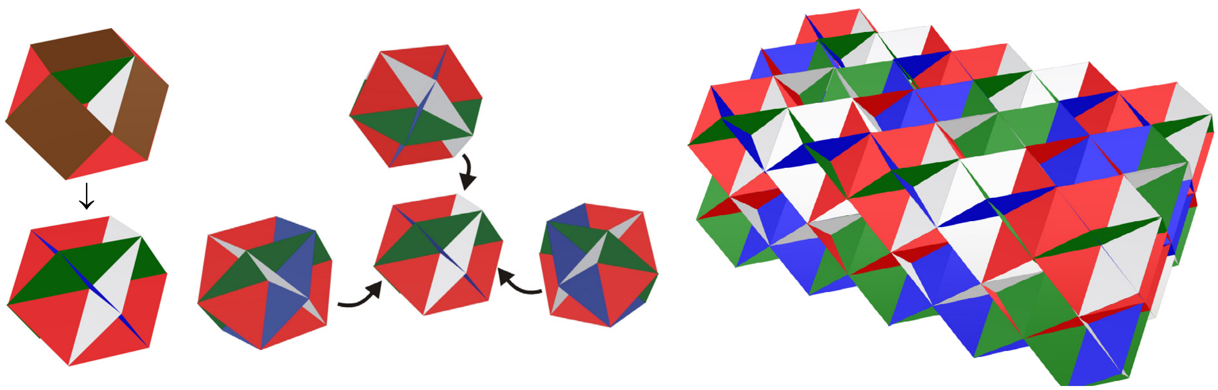

Figure 2: *Based on adjacent open cubohemioctahedra (left), a regular infinite Kepler-Poinsot polyhedron {6, 8} is built (right).*

The polyhedron is regular just as the 5 Platonic solids, but has intersecting faces, like the 4 regular Kepler-Poinsot solids, and it is infinite, like the 3 regular Petrie-Coxeter polyhedra (see Fig. 1). Thus, it is an example of a polyhedron that merges the Kepler-Poinsot and the Petrie-Coxeter ideas and perhaps that is why it remained undiscovered ever since (see [1]). One could discuss that really is a *new polyhedron,* because it is in fact a compound, just as the Kepler star is a compound of two tetrahedra, but it deserves its title of polyhedron just as the Kepler star does. The formula $aV + bF - E = 2c(1 - g)$ is now verified for the {6, 8} case as $a = 1$, $b = 1$, $c = 2$, $g = 3$ and thus $24 + 16 - 48 = -8 = 2 \times 2 \times (1-3)$.

As the {6, 8} is a regular polyhedron, its dual, an {8, 6}, should provide yet another new regular polyhedron, in which six octagons meet in each vertex. However, the dual of the cubohemioctahedra, on which the building blocks of the {6, 8} are based, is degenerate. It is the hexahemioctacron, which Magnus Wenninger represented by intersecting infinite prisms passing through its centre (see [9]). Still, as two layers of the {6, 8} can be seen as a compound of four Petrie-Coxeter {6, 4} polyhedra, it was suggested to consider four adjacent {4, 6}s as the dual of two layers of {6, 8}s. The cubes are stacked on each other's faces, face to face, so that the faces count double. Thus, the dual polyhedron is made out of squares that are counted twice, which can be interpreted as special octagonal (star) polygons, like the 8/3 and an 8/2 star octagons (see Fig. 3). With this interpretation, twelve 'double-square-octagons' meet in each vertex of the dual of two layers of the {6, 8}. Thus, one layer of the dual of the {6, 8} is formed by 6 'double-square-octagons' in each vertex, and thus indeed by an {8, 6}. This is indeed a regular infinite polyhedron, though an odd one.

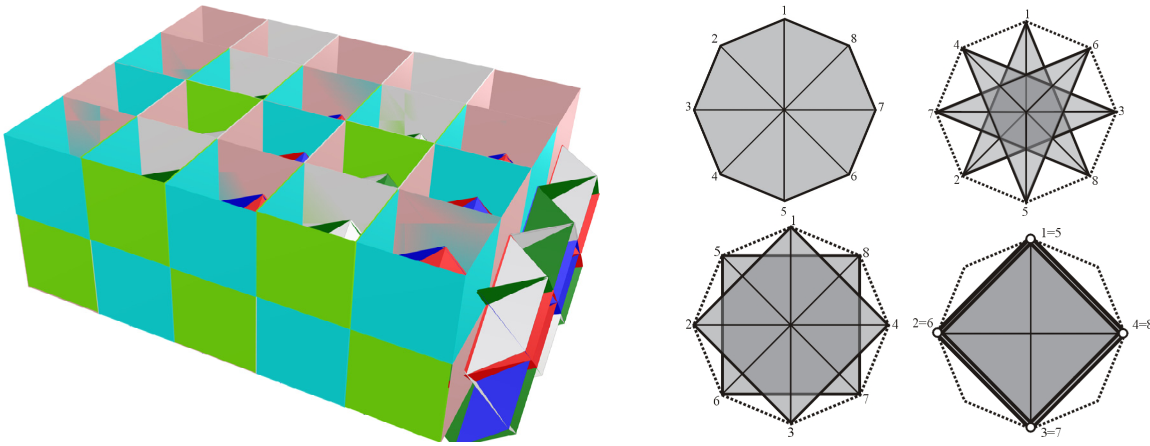

Figure 3: *The {6, 8} and its dual, the {8, 6}, formed by adjacent {4,6}s (left); a regular 'standard' octagon, 8/3 and 8/2 star octagons and a 'double square octagon' (right).*

Another way to get a better understanding of this regular infinite Kepler-Poinsot polyhedron {6, 8} was obtained by introducing prismatic tunnels (open cubes) between the open cubohemioctahedra. This turns the {6, 8} in an 'Archimedean type' infinite Kepler-Poinsot polyhedron {4, 6, 4, 4, 6, 4}, that is, a polyhedron composed from two types of regular polygons, in this case, hexagons and squares (see fig. 4).

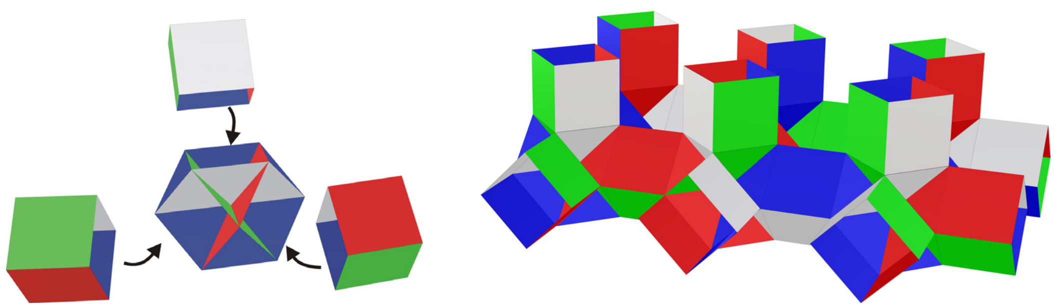

Figure 4: *An infinite Kepler-Poinsot polyhedron 'of Archimedean type': the 'regular {6, 8}' with open cube tunnels, a {4, 6, 4, 4, 6, 4}.*

An example with tetrahedral symmetry

In a similar way, one could be tempted to suggest yet another new regular (compound) polyhedron starting from the tetrahemihexahedron or hemicuboctahedron. This is a uniform star polyhedron formed by four equilateral triangles and three squares, in the same vertex and edge configurations as the regular octahedron. It shares 4 of the 8 triangles of such an octahedron, but has three square faces passing through its centre.

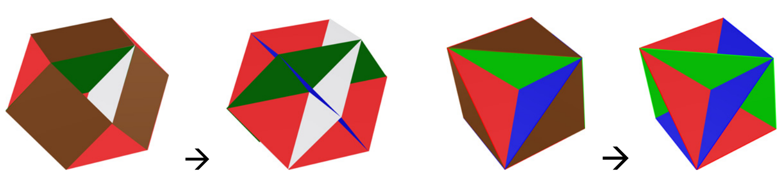

Figure 5: *The cubohemioctahedra and its open version (left), and the tetrahemihexahedron with its open version(right).*

Removing the triangles an open tetrahemihexahedron remains, with only 4 squares as faces, going through its centre. Adjacent open tetrahemihexahedra yield an infinite 'polyhedron', in which 12 squares meet in each vertex. Thus it is a {4, 12}, and it is a regular structure, as all faces are square meeting in the same spatial angle.

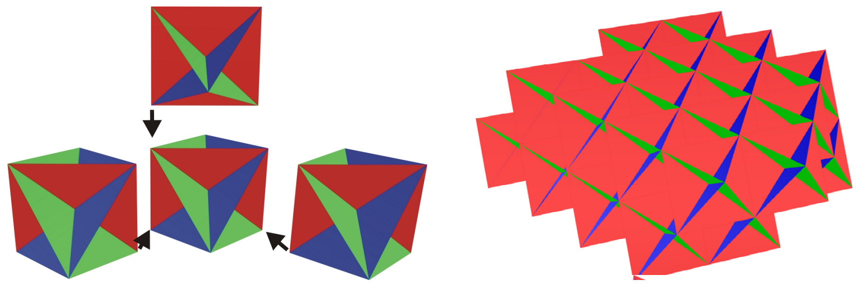

Figure 6: *Joining tetrahemihexahedra to make an another new regular 'polyhedron', a {4, 12}.*

For the tetrahemihexahedron, $V = 6$, $F = 7$, $E = 12$, so that an open version corresponds to $V = 3$, $F = 3$, $E = 6$. The formula becomes $3 + 3 - 6 = 0 = 2\times c\times(1-1)$ (the value of c does even matter). An interpretation is that this 'infinite polyhedron' can be seen as the intersection of 3 planes formed by the parallel square faces. Indeed, putting the tetrahemihexahedra next to each other, creates an infinite polyhedron with coplanar adjacent faces. Many authors, such as Coxeter, do not consider such a structure as a polyhedron, but others do, such as Gott (see [3]).

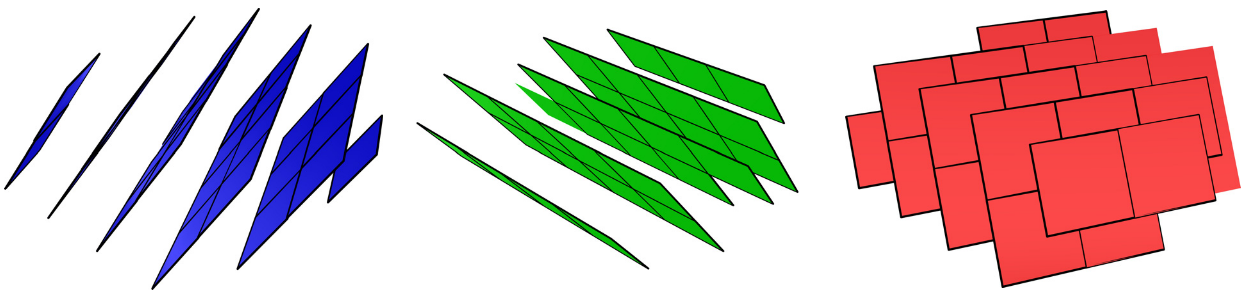

Figure 7: *Placing tetrahemihexahedra next to each other creates a structure of 3 infinite families of parallel planes.*

Again, the construction dual is not obvious, since the dual of the tetrahemihexahedron is the tetrahemihexacron with vertices at infinity. However, if we again count the centres of the three squares going through the same point as a triple point, the duals consist of squares formed by triple counted points. They can be seen as special dodecagons. Thus, the dual of the {4, 12} would be a {12, 4}.

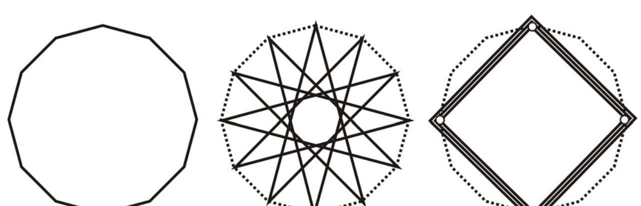

Figure 8: *A regular dodecagon, a 12/5 star dodecagon, and a triple square 'dodecagon'.*

To get a more spatial view on the polyhedron, we again can add tunnels, such as open octagons (octagons missing two parallel faces, or triangular anti-prisms; see Fig. 9). It again is an infinite Kepler-Poinsot polyhedron 'of Archimedean type'. Its symbol is {4, 3, 3, 3, 4, 3, 3, 3}.

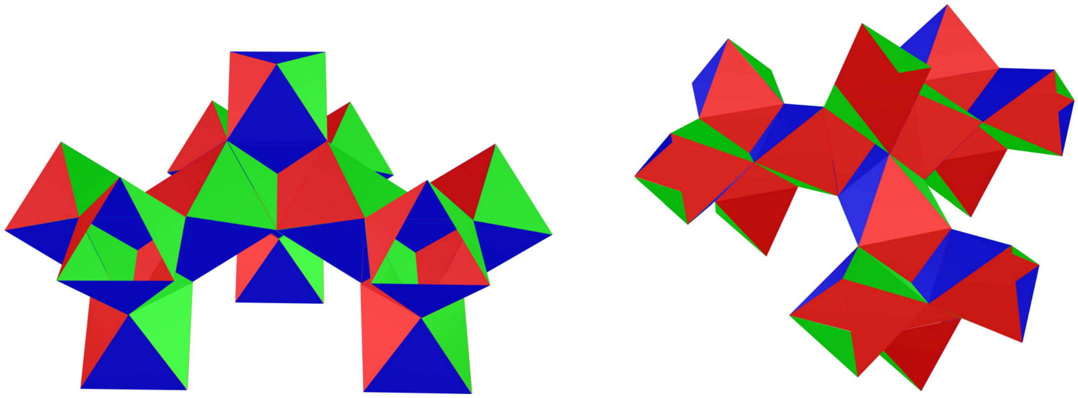

Figure 9: *Two views on the infinite open tetrahemihexahedra polyhedron with triangular anti-prism tunnels.*

An example with icosahedral symmetry

Icosahedra can be connected by octahedral tunnels to create an infinite polyhedron with seven triangles meeting at each vertex. It has a six-membered ring repeat unit and this is said to be a model for carbon bonds in diamond crystal (see [2]). However, we can also replace the regular icosahedra by their star shaped equivalents, that is, by great icosahedra. Many faces of the octahedral tunnels now intersect but this is allowed in a Kepler-Poinsot construction. Of each icosahedron, 4 (triangular) faces are removed, on which the open octahedral tunnels will be placed. We proceed similarly with the great icosahedra: we remove each time four of their 20 (triangular) faces. It is more difficult to see, as this removal doesn't create any openings.

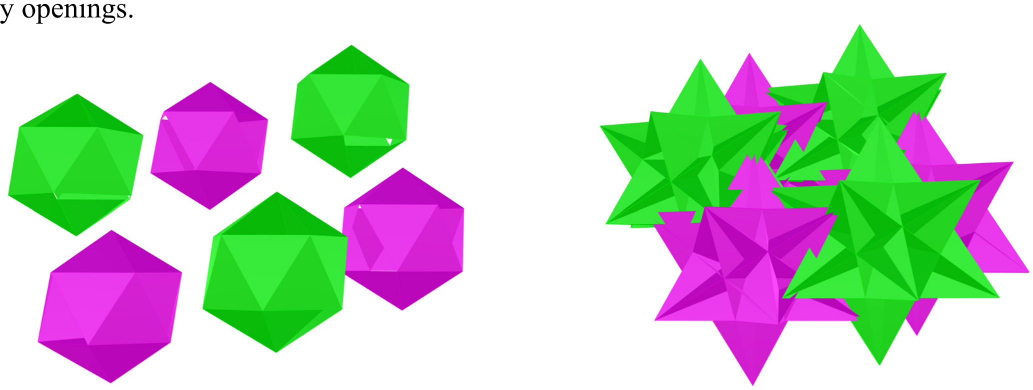

Figure 10: *Six icosahedra without four of their faces, and six great icosahedra without four of their faces.*

Next, we place the open octahedral tunnels between them (see Fig. 11). Those on the great icosahedra intersect at many occasions so that it is harder to see they form a ring (see Fig. 13).

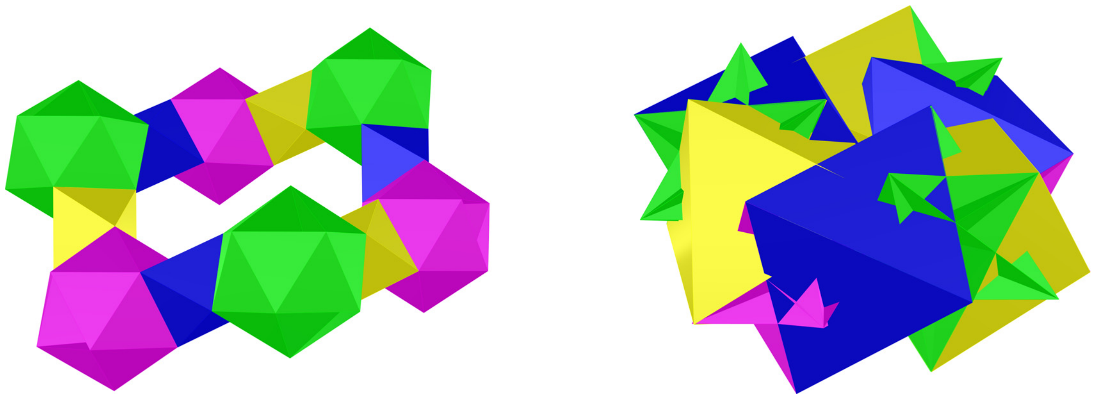

Figure 11: *Six icosahedra connected by tunnels, and six great icosahedra connected by tunnels.*

The construction is completed with open octahedral tunnels on top and bottom, so that it can be continued below and above.

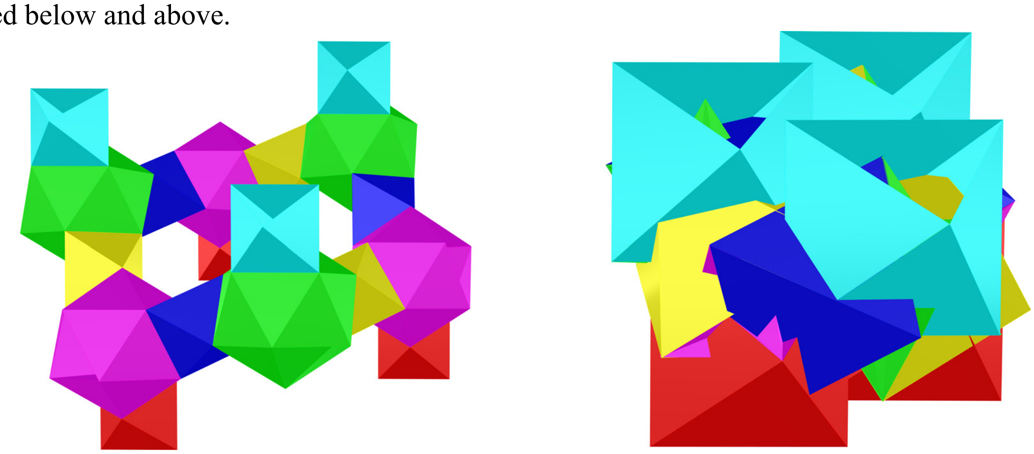

Figure 12: *The completed six-membered ring of open icosahedra with octahedral tunnels and the completed six-membered ring of open great icosahedra with octahedral tunnels.*

Fig. 13 also shows the building blocks for both the ring of open icosahedra and open great icosahedra: in each case, there are two enantiomers. The Euler formula $V + F - E = 2(1 - g)$ is readily verified for the first case, as $V = 12$, $F = 16 + 12 = 28$ and $E = 30 + 12 = 42$: $12 + (16 + 12) - (30 + 12) = -2 = 2\times(1-2)$ (see [5]). The second case however is not straightforward. The (regular, closed) great icosahedron shares the same number of vertices, faces and edges with the icosahedron, but its density is 7, as a line through its centre crosses the polyhedron 7 times. The vertices have density 2 and the Cayley-Euler formula is $2\times12 + 20 - 30 = 14 = 2\times7$. After the removal of 4 of its faces, the density changes, and moreover the tunnels have density 1. Thus, it doesn't seem possible to get a straightforward application of the 'generalized Cayley-Euler formula' – perhaps it has restricted validity conditions.

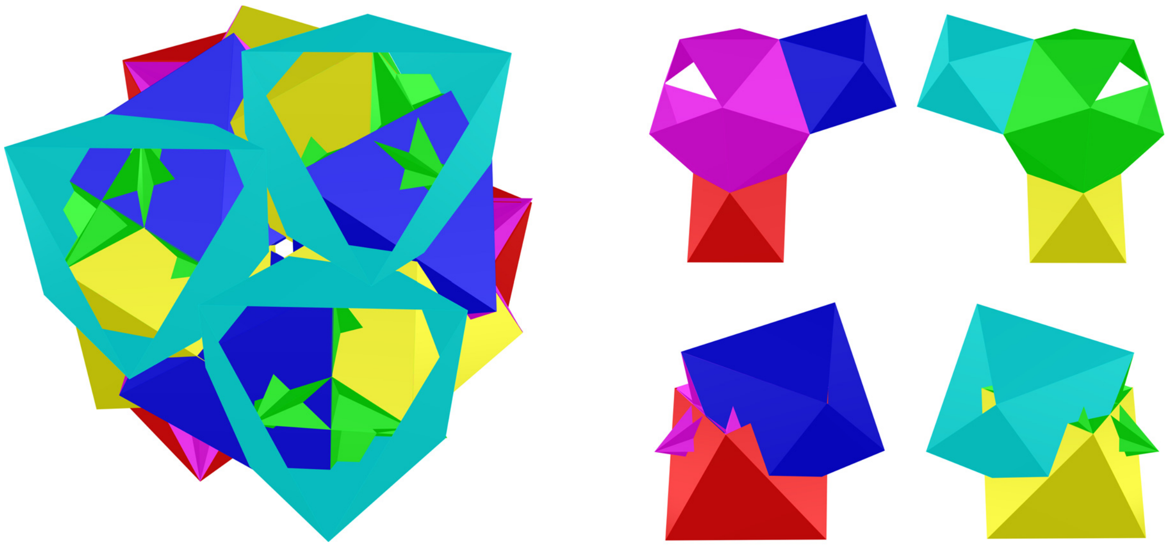

Figure 13: *Top view on the ring of great icosahedra with octahedral tunnels, and a comparison of the enantiomer building blocks in each case.*

Artistic applications

As a referee pointed out, polyhedra have an intrinsic beauty. We now try to emphasize this by presenting more images of the polyhedra given above. First follows an adaption of the octahedral symmetry example.

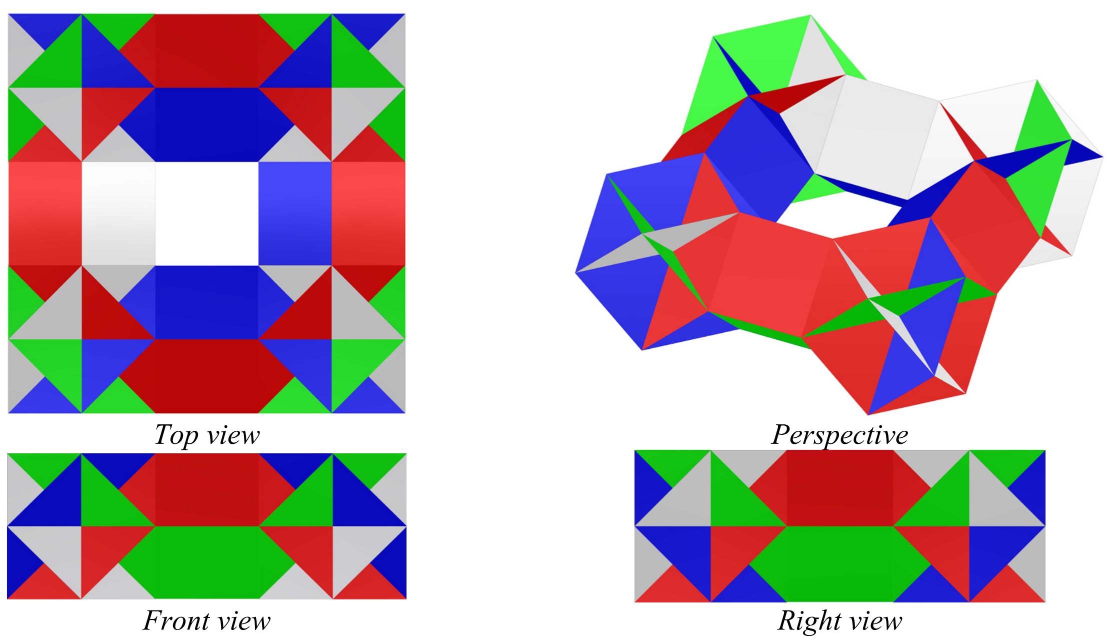

Figure 14: *More views on the infinite Kepler-Poinsot polyhedron {4, 6, 4, 4, 6, 4}.*

Next, we repeat the tetrahedral symmetry example {4, 3, 3, 3, 4, 3, 3, 3} given above, but with more elements.

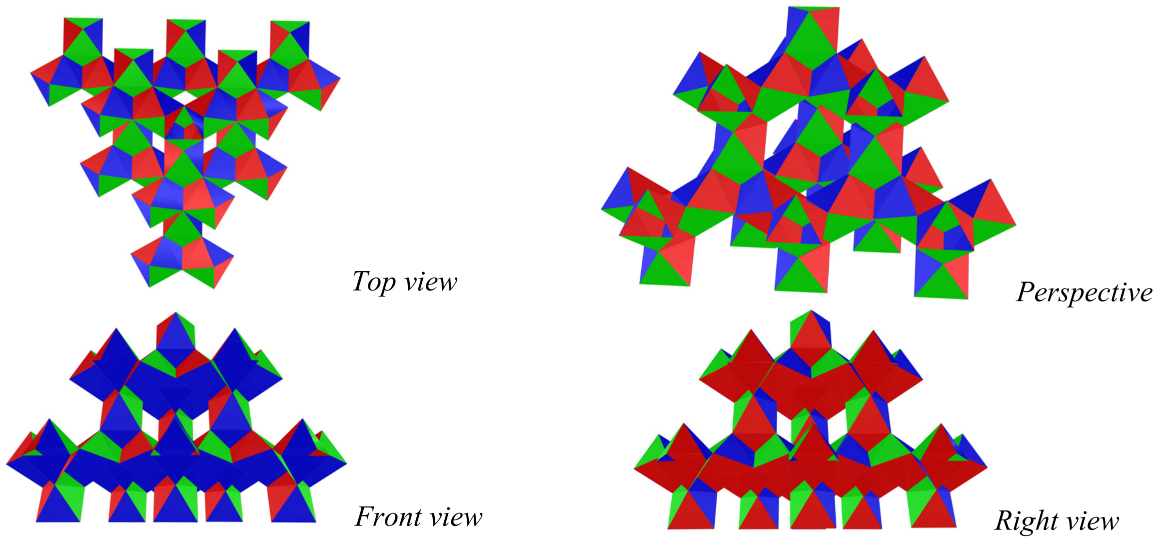

Figure 15: *More views on the infinite Kepler-Poinsot polyhedron {4, 3, 3, 3, 4, 3, 3, 3}.*

The example with icosahedral symmetry seemed more technical and hard to grasp with its overload of intersection faces, so here are two more artistic representations, identical to Fig. 12b.

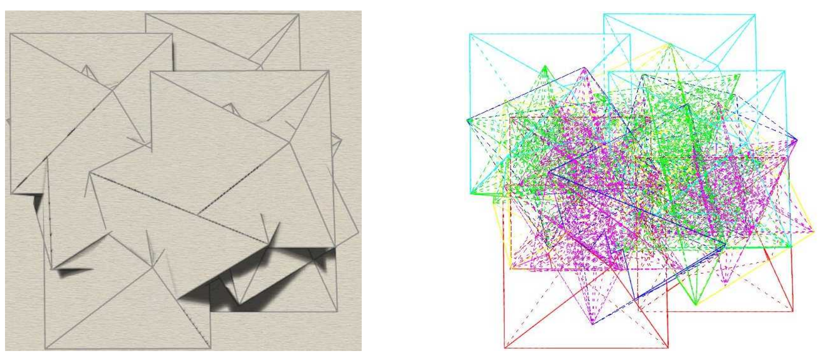

Figure 16: *More representation of the icosahedral case.*

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