Fractal Islamic Geometric Patterns Based on Arrangements of $\{n/2\}$ Stars

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Abstract

Within traditional Islamic geometric design, there is a small but distinct subset of patterns which incorporate notions of self-similarity. Such patterns typically use exactly two overlapping levels of self-similarity, with each level comprising similar motifs at two different, related scales. Recently I have constructed a large number of fractal Islamic patterns that pursue a different notion of self-similarity, simultaneously incorporating motifs at multiple—often, infinite—scales within a single pattern. I will first highlight the ways in which these patterns are distinct in nature from related work, both traditional and contemporary. Then, I will explain the details of the technique I have devised for using arrangements of $\{n/2\}$ stars as a basis for discovering and constructing a wide variety of new fractal Islamic patterns.

Introduction

Islamic geometric design is a well-known artistic medium dating back many centuries. In recent years many artists, myself included, have been inspired to create new works which build off of the patterns and motifs found in these historical works. Historical research has shed light on how such patterns were originally constructed, and modern mathematical and computing techniques have allowed for the extension of those methods to create endless new variations.

Related Work: A Brief Survey of Pre-Existing Self-Similar Islamic Designs

In his Bridges 2003 paper, Jay Bonner [2] gives a thorough description of three types of self-similar geometric design that appear in the historical record, which he there labels A, B, and C. (He has recently described a fourth variation as well [4]). These pattern types, shown in Fig. 1, share two notable qualities despite their varied appearance. First, there are typically only two levels of pattern—perhaps due to limitations of technology at the time. Second, the two levels or sizes of motif occur simultaneously, one on top of the other.

One contemporary method being used by Marc Pelletier [15] and Jean-Marc Castera [6] involves taking the historical types mentioned above, analyzing their construction, and using those methods combined with the power of modern computer software to construct new patterns in the same style but with three or more levels of recursion (Fig. 2, left). This method yields beautiful new patterns that share many qualities with their historical antecedents. Another modern method, described in detail by Kaplan and Salesin [12], involves describing patterns in a way that avoids use of the parallel postulate, thus allowing them to be projected into Euclidean, spherical, and hyperbolic spaces (Fig. 2, center). Hyperbolic patterns have also been explored by Dunham [7]. The hyperbolic case leads to patterns with infinite simultaneous sizes of the basic motif; however, the motifs are necessarily distorted when represented in the Euclidean plane. A third avenue of exploration is being conducted by Joe Bartholomew [1]. He uses a variation on the traditional "Girih" tiling technique, combined with scaling techniques of his own creation, to produce art with undistorted motifs at many different scales (Fig. 2, right). Bonner has also experimented with similar techniques [4]. Notably, Bartholomew often employs pseudo-random techniques to purposely avoid a high degree of symmetry in his final works.

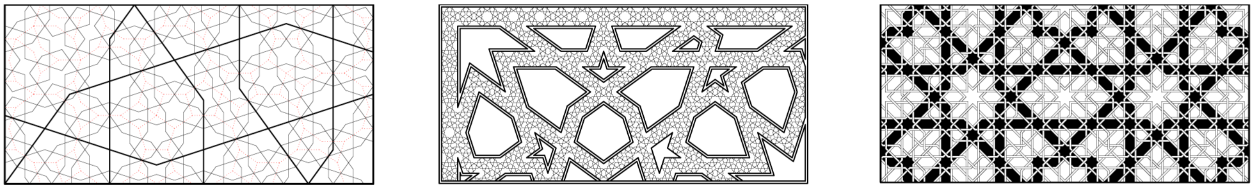

Figure 1 : *Left to right: traditional self-similar pattern types A, B, and C (per Bonner).*

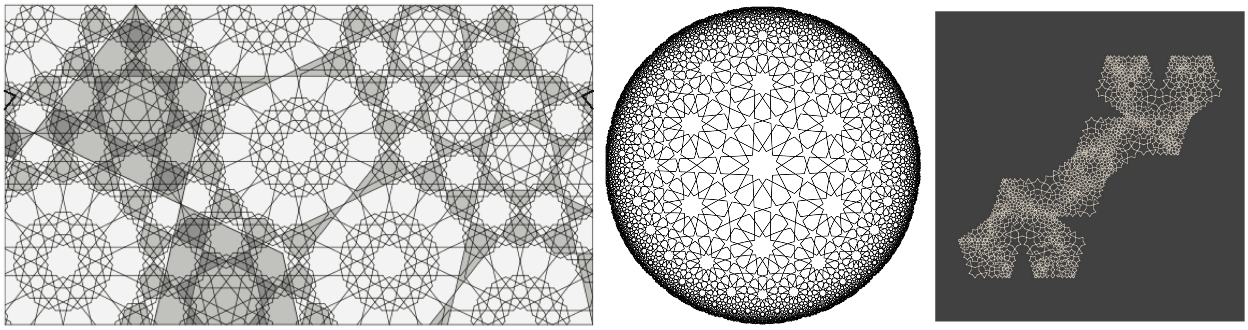

Figure 2 : *Left to right: works by M. Pelletier, C. Kaplan, and J. Bartholomew (used by permission)*

A New Kind of Fractal Islamic Pattern

While bearing resemblances to some of the works mentioned above, the patterns discussed here are different in both their method of construction and in certain key qualities of their final form. These differences stem primarily from my own artistic choices and aesthetic inclinations, and can be summarized as follows:

- **Multi-scale but single-level.** Unlike traditional works [5] and those of Bonner and Pelletier [15] and Castera [6], my patterns include multiple sizes of motif all at a single level, rather than two or more levels superimposed on top of one another.
- **Motifs at more than two scales.** Also unlike traditional designs (though similarly to many contemporaries), these patterns include motifs at three or more—sometimes infinite—scales.
- **Undistorted motifs.** Unlike hyperbolic designs like those of Kaplan and Salesin [12] or Dunham [7], the motifs in these patterns are always undistorted.
- **Regular, symmetric, and interlaceable.** Finally, these patterns are designed to be highly regular, symmetric, and capable of being decorated in all of the traditional ways, including the *interlaceable* style—all differences from most of Bartholomew's work.

Use of Star Subgrids vs. Polygon Subgrids

The majority of creators of modern Islamic patterns use traditionally derived methods based on subgrids of polygons called variously "polygons-in-contact" [13],"girih tiles" [14], "generative subgrids" [3], or "inflation tilings" [12]. In these methods, an underlying subgrid of polygons is created and then the polygons are filled with motifs or, alternatively, sets of crossing lines are added at the polygon midpoints. The grid is then removed, leaving the final pattern. I am currently working on "reverse engineering" my patterns to see how they relate to these methods. However, being unaware of them when I embarked on this investigation, I used a different method of my own creation based on patterns of $\{n/2\}$ stars.

Stars as motif placeholders. Using the convention of Grünbaum and Shephard [11], a star polygon $\{n/d\}$

is formed by joining each corner of a regular n-gon $\{n\}$ to the d^{th} corner in each direction. A few examples are shown on the left in Fig. 3. My starting point was the realization that an $\{n/d\}$ star could serve as a "placeholder" for any of the three most prominent historical Islamic star motifs, which I will refer to as stars, rosettes, and extended rosettes, as pictured on the right in Fig. 3.

Figure 3: *Examples of* $\{$*n/d*$\}$ *stars and traditional Islamic star motifs*

Star-to-star configurations. There are two obvious ways to arrange two $\{n/d\}$ stars with respect to each other. The first, shown in Fig. 4(a), I call *single point*, where the stars touch point-to-point with the stars' centers forming a line through the point of contact. The second, shown in Fig. 4(b), I call *two point*, where the stars touch at two consecutive points, with the centers forming a line bisecting the generating n-gon's side lying between the two points. Any of the $\{n/d\}$ stars, including n-gons themselves, can be arranged equally well in these two configurations.

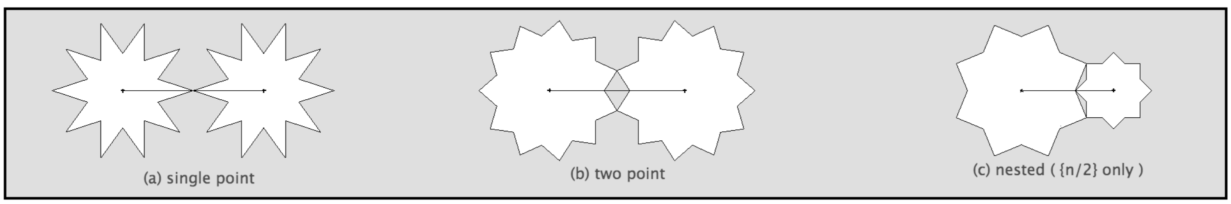

Figure 4: *Star-to-star configurations*

Properties of each configuration. After exploring the first two configurations, I discovered that each naturally leads to different sorts of star patterns. The two point case assures that all stars are of the same scale, so there is no case of a purely two point star grid that will have any fractal nature. In the single point case, one is free to scale the stars to any degree desired. As shown in Fig. 5, I discovered many beautiful periodic tessellations based on two sizes of star, and a handful with three sizes, but there was no obvious way to construct patterns with larger or infinite scales simultaneously.

My eventual selection of $\{n/2\}$ stars hinges on a special property that other $\{n/d\}$ stars lack: the reflex angles (or "dents") of an $\{n/2\}$ star always have the same angle measure (namely, $\theta = \pi\frac{(n-2)}{n}$) as the exterior angle of the corresponding n-gon. This allows for a third configuration I call *nested*, shown in Fig. 4(c), where three points of one star contact two points and a dent of another star. With this configuration, stars of different scales are assured by the nature of the configuration—the scale factor being $\frac{1}{2sin(\theta/2)}$ where θ is defined as above. The rest of this paper discusses the patterns that result from this nested case.

Arrangements of $\{n/2\}$ Stars

If one begins with an $\{n/2\}$ star ("level 0") with $n \geq 6$ and n even (the case of odd n is considered in "Future Directions" below), one can then arrange n stars ("level 1") around its perimeter using the nested configuration. This procedure can then be repeated around the unoccupied dents of the Level 1 stars to produce Level 2, and so on *ad infinitum*. For $n \geq 10$ the stars begin to overlap, but always in such a way that

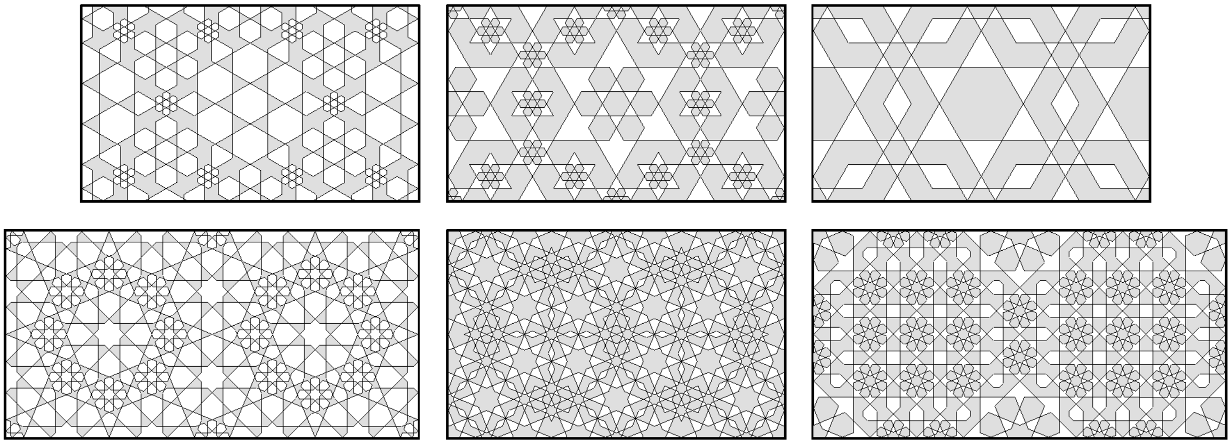

Figure 5 : *Some interesting, but not truly fractal, patterns based on single point star configurations*

a 3-dent gap remains between them so that the tiling can still be carried out in a sensible way, and can still ultimately lead to usable patterns. Fig. 6 shows these arrangements for the first few values of n.

Ultimately I came to realize that a key property of these arrangements was the positions of the centers of the stars. If one connects the center of each star on one level to the centers of its connected stars on the next level, one reveals an underlying framework, as shown in Fig. 7, which is itself formed entirely of $1/n$ wedges of $\{n/d\}$ stars, with $d = (n/2) - 2$. These bear a strong resemblance to many of Fathauer's fractal tilings formed from $1/n$ wedges of n-gons (or "kites") [8],[9]. Also, at least one of the star grids ($n = 6$) corresponds exactly to an unpublished tiling by Fathauer [10].

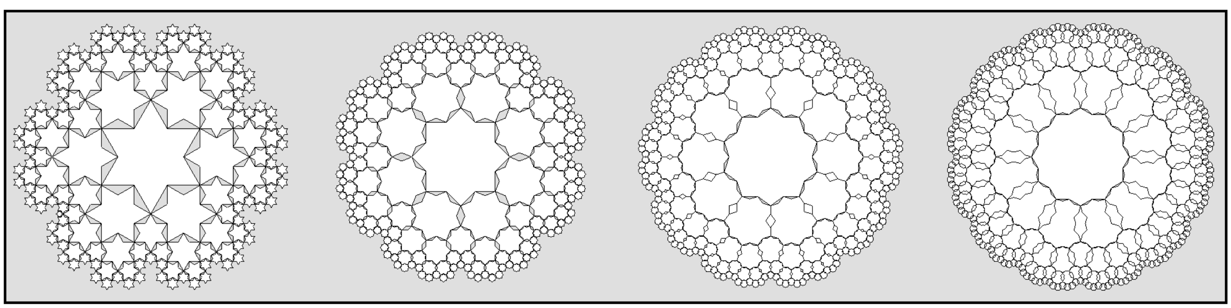

Figure 6 : *Left to right: Radial nested star patterns for* $n = 6$*,* $n = 8$*,* $n = 10$*, and* $n = 12$

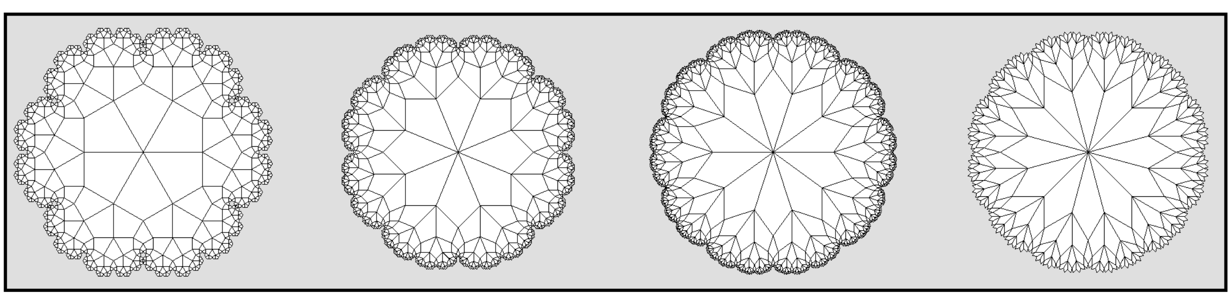

Figure 7 : *Underlying frameworks formed by the centers of the stars shown in Fig. 6*

As illustrated in Fig. 8, one wonderful outcome of this underlying structure is that each *outward* radial arrangement has a corresponding *inward* variant. Furthermore, since both arrangements share the same ring of n stars (level 1 of the *outward* grid), they can also be joined into a *combined* arrangement, analogous to Fathauer's "perforated" tilings [9], and also reminiscent of M.C. Escher's 1969 print *Snakes*. Finally, these various radial configurations can, depending on n, often be further combined into *repeating* plane tessellation arrangements, alternating inward and outward scaling. Since these arrangements share the same level-to-level scale factor and structure, once one has been transformed into an Islamic pattern (as described next), the others can be transformed in exactly the same way.

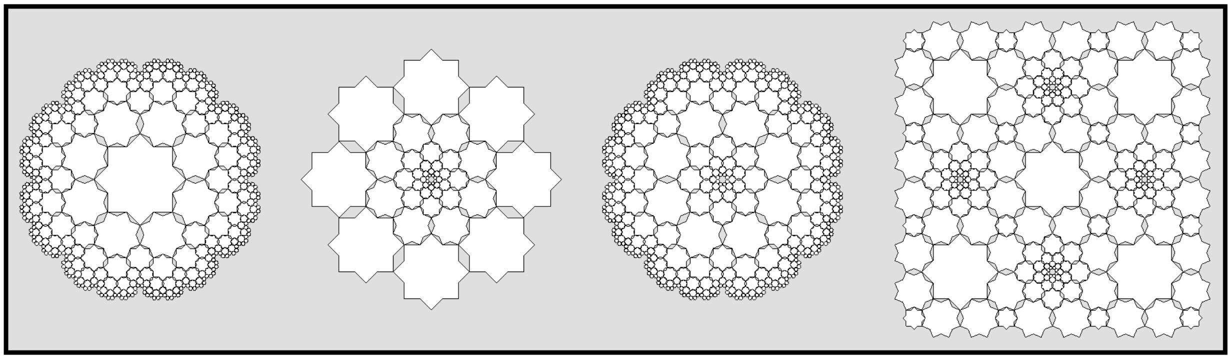

Figure 8 : *Star arrangement types for $n = 8$ (left to right): outward, inward, combined, repeating*

Transforming an $\{n/2\}$ Star Arrangement Into an Islamic Pattern

For any given star arrangement, I go through a series of steps to generate usable, pleasing Islamic patterns based on that arrangement. First, I substitute each of the three basic motifs (star, rosette, and extended rosette) for every $\{n/2\}$ star, yielding three base patterns. If the motifs don't already align nicely, I next transform motifs via scaling, rotation, and/or variation of angles at the points to improve the alignments, as illustrated in Fig. 9. Finally, I add interstitial figures to complete the pattern, and then select the most pleasing patterns and decorate them to produce final art pieces. This whole process is visualized for the case of $n = 8$ and the rosette motif in Fig. 10.

The parts of the process with the most latitude are the transformation of the motifs and the addition of interstitial figures. The most common transformation I perform on the motifs is simple scaling about the center in order to eliminate overlap, make points of adjacent motifs coincident, and/or create other useful alignments between motifs (Fig. 9, left). Occasionally I will also rotate motifs by π/n degrees, again to make adjacent points coincident (Fig. 9, center). Finally, in some cases, I widen or narrow the angles at the points (especially of the star motif) in order to create a smoother line flow from one level to the next (Fig. 9, right). When adding interstitial figures, my goal is always to achieve visual balance and create shapes that are as consonant with the historic record as possible. This is a trial-and-error process based on intuition and familiarity with traditional patterns.

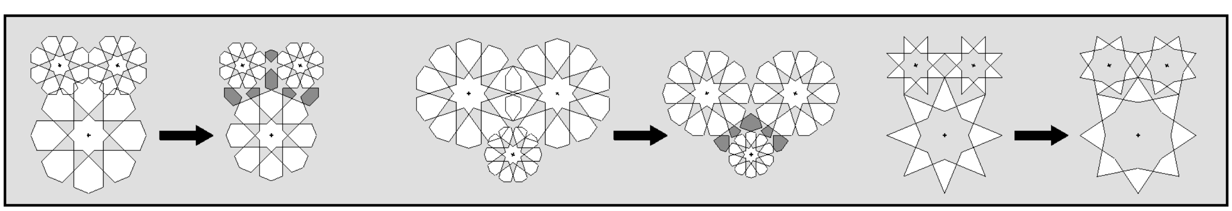

Figure 9 : *Motif transformations (left to right): scaling, scaling plus rotation, and change of point angles [added interstitial figures shown in dark grey]*

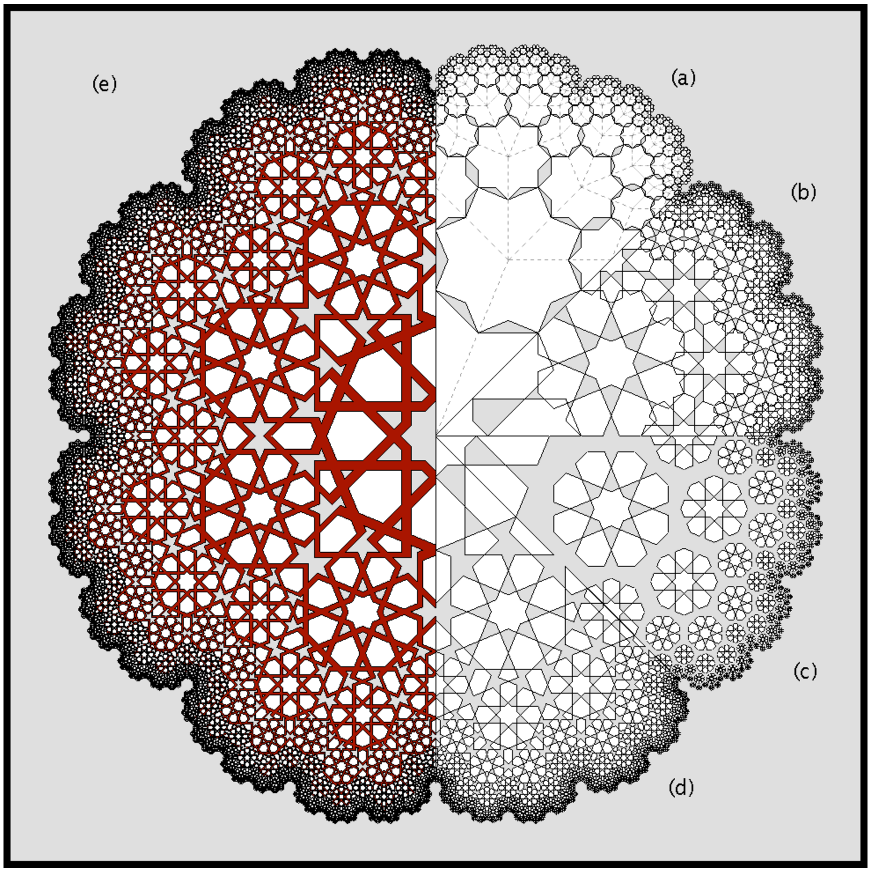

Figure 10 : *Anatomy of a fractal Islamic pattern: (a) $\{n/2\}$ star arrangement (8-fold outward); (b) motif replacement (rosettes); (c) motif modification (scaling); (d) addition of interstitial lines; (e) final decorated pattern (outline with variable band width + two-color mosaic)*

Decoration Considerations

Once a final pattern has been created, decoration can in many ways proceed exactly as with traditional patterns, including (amongst others) a *mosaic* treatment where shapes are filled in with two or more colors; an *outline* treatment where lines are thickened and/or multiplied; and an *interlace* treatment where the thickened lines are drawn as if to wind over and under each other. However, two aspects of decoration are unique to the fractal nature of the designs.

In traditional *mosaic* treatments, since the pattern typically tiles the plane, once a color scheme is chosen it simply repeats. With radial arrangements, however, there is the opportunity to use color to emphasize the levels of the design, as in Fig. 11(a). When decorating in the *outline* and *interlace* styles, a new challenge is faced: how does one handle motifs that diminish to infinity? There are two options. If a uniform band width is chosen, only a few levels can be depicted before the bands collide. However, the relative shift of the ratio between band width and motif size from level to level can produce interesting results—e.g. the different characters of the three levels of rosettes in Fig. 11(b). If, on the other hand, variable band width is used, then the pattern can be carried out down to the available resolution for reproducing the design, as in Fig. 11(c). This sort of variable band width has appeared previously in works by Fathauer and Kaplan. Combining all of these techniques in different combinations leads to a nearly endless array of possibilities. Some examples of art work created using various combinations of choices are shown in Fig. 12.

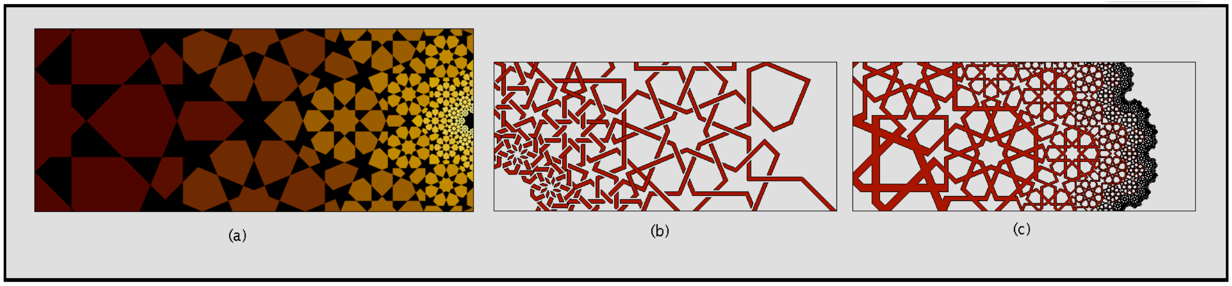

Figure 11 : *Decoration options: (a) color by level; (b) equal band width; (c) variable band width*

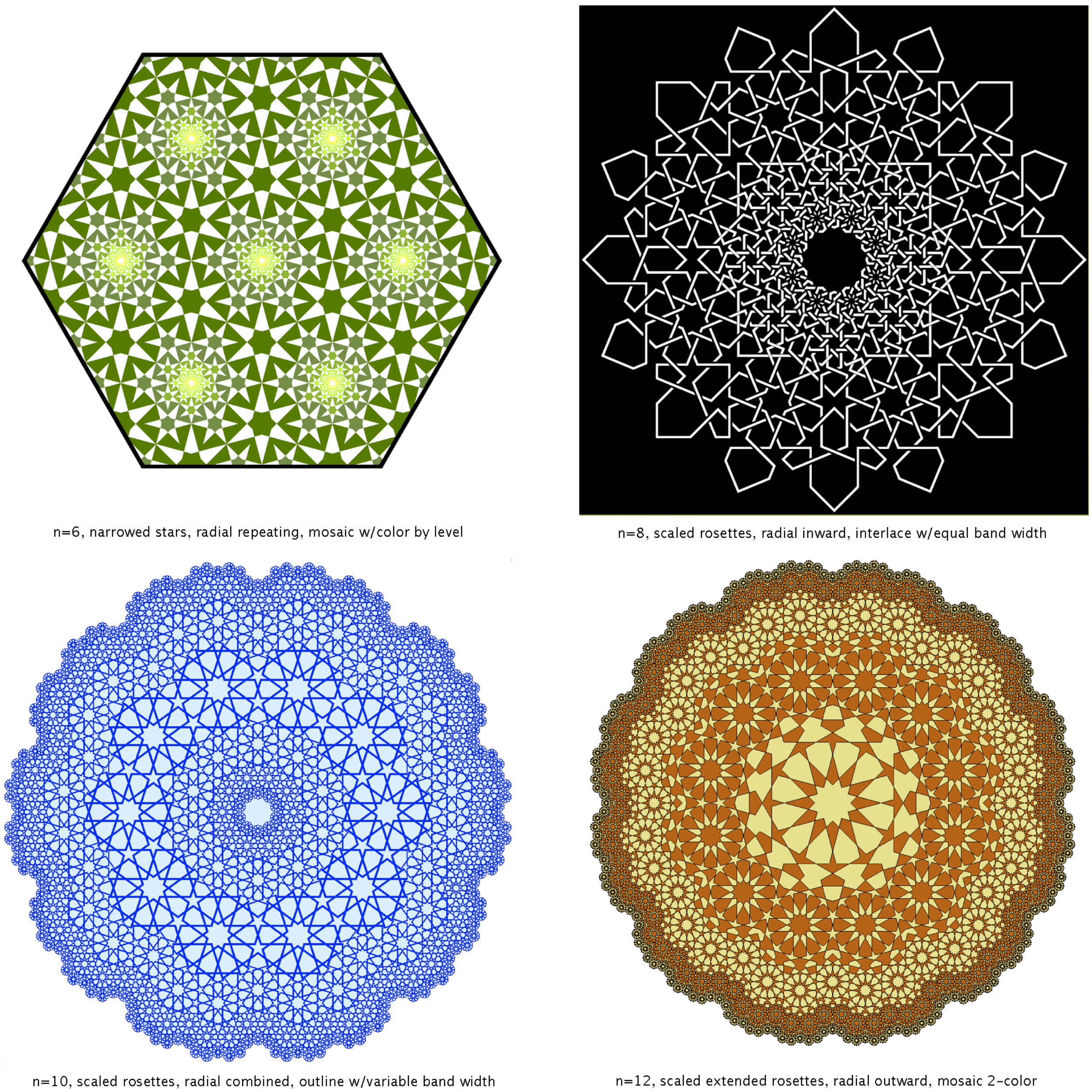

Figure 12 : *Examples of artwork based on the techniques described here. Captions list the value of* n*, motif and transformation choices, underlying star arrangement, and decoration style*

Future Directions

Despite the large variety of patterns already presented here, I have many more directions yet to pursue. At this stage I create all the patterns described here essentially "by hand" using modeling and illustration software, but a clear opportunity exists to automate some or all of the pattern generation steps algorithmically. As previously mentioned, I also plan to examine these patterns in light of the "generative subgrid" approach, to see if this might lead to further avenues of exploration. This includes the obvious step of widening the range of motifs; right now I have been using (in Bonner's terminology [3]) the *acute* versions of the various motifs (except where star scaling has been used), and I'm eager to explore *median*, *obtuse*, and *two-point* variations. Also, although my initial explorations using odd values of n did not work as well as the even cases discussed here, I intend to explore them further to see if they might still yield usable patterns. Finally, the process of motif scaling leads to entire families of patterns for a given underlying star arrangement and motif choice, varying only by the relative size of the motifs and the resulting interstitial shapes. These can potentially be combined into animations showing these patterns gradually morphing into each other.

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