The Pentagram: From the Goddess to Symplectic Geometry

Elisa Prato
Dipartimento di Matematica e
Applicazioni per l'Architettura
Università di Firenze
Piazza Ghiberti 27
50122 Firenze, ITALY
elisa.prato@unifi.it

Abstract

In this paper we show that the pentagram is a natural ingredient of two seemingly unrelated areas of knowledge: the divine feminine, and the symplectic geometry of Penrose rhombus tilings.

Introduction

This paper is an account of the author's personal journey, from the day that we discovered that two, seemingly unrelated, fundamental interests of ours, a personal interest in the divine feminine, and a scientific interest in symplectic geometry, turned out to be part of the same larger scheme, like two different shores of the same river.

The bridge between these two shores is provided by the pentagram, which happens to occur naturally in both areas. In fact the pentagram is a very important ingredient in the author's joint work with F. Battaglia on Penrose rhombus tilings from a symplectic viewpoint [3]. While this work was going on we learned from chapter 6 of D. Brown's "da Vinci code" [4] that the pentagram was also an ancient symbol of the divine feminine.

The author's journey is presently far from being complete, and though it is quite obvious that the occurrence of the pentagram in two of our main areas of interest is no coincidence, we still do not fully understand its potential benefits to either area of knowledge. We plan to address this in future work.

The article is structured in the following way. In the first section we attempt to clarify D. Brown's statement above by describing the importance of the pentagram in the Babylonian and Greek female pantheons. In the second section, following [3], we discuss the relevance of the pentagram in the study of the symplectic geometry of Penrose rhombus tilings.

1 The Pentagram and the Goddess

It is certain that the pentagram comes from Babylon. It is first found in the Uruk IV period (ca. 3300-3100BC) as a Sumeric pictographic symbol. Its exact meaning at that time is unclear. In later cuneiform texts (ca. 2600BC) it is the Sumeric sign UB, whose meaning seems to be "heavenly region" (see Appendix

A in de Vogel [5]). According to some (see for example a reference to Fritz Röck in [5], Appendix A), Babylonian priests identified the pentagram with Ishtar, the goddess of love and war in the Babylonian pantheon. Ishtar derived from the Sumeric goddess Inanna, and both were divine representations of the planet Venus. The reason for the association of Ishtar with the pentagram is still undocumented, but what appears to be a strong clue is the astronomical pattern of the Venus/Sun conjunctions. A superior conjunction occurs when Venus is behind the Sun and an inferior conjunction occurs when Venus is between the Earth and the Sun. If one traces the position of Venus on the zodiac for each superior conjunction over a period of eight years one obtains an almost perfect pentagram. The same thing is true for the inferior conjunctions. This fact was certainly known by Babylonian astrologists at least as early as the 17th century BC: in fact it can be deduced from data in the Venus tablet of Ammisaduqa, a 7th century BC cuneiform document that relates knowledge that dates back to the reign of king Ammisaduqa (1646-1626BC). Reiner and Pingree [10] argue that, though this is the first known written document on the matter, it almost certainly incorporates elements of a longstanding tradition. Another interesting link between the pentagram and Ishtar is that the first pictographic documents containing the pentagram were found in the city of Uruk, which was the major worship center for the goddess.

Centuries later, the Greek mathematician Pythagoras learned of the pentagram during his stay in Babylon (ca. 554–533BC). The Pythagoreans adopted the pentagram as their distinctive symbol and associated it with the goddess of health, Hygeia. The reason for this association is unclear, but it is possibly a consequence of the Babylonian association of the pentagram with the goddess Ishtar. The Pythagoreans labeled the vertices of the pentagram, after Hygeia, with the Greek letters $Y - \Gamma - \mathrm{I} - \mathrm{EI} - \mathrm{A}$ (see Figure 1). These letters were later interpreted to symbolize the elements, since they are the initials of the Greek words that translate to water, earth, divine/holy thing, warmth/fire, air.

2 The Pentagram and Symplectic Geometry

The focus of this section is to describe the occurrence of the pentagram in the author's joint work with F. Battaglia [3] on Penrose rhombus tilings from a symplectic standpoint.

We begin by outlining a procedure for obtaining the Penrose rhombuses from the pentagram. One of the main ingredients of this procedure is the *golden ratio*: $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$. For the properties of this number and for the proofs of the facts that follow we refer the reader to the beautiful book by M. Livio [8]. Let us inscribe the pentagram in a regular pentagon, as in Figure 1. It can be shown that the ratio of the diagonal to the side of the pentagon is equal to ϕ. Therefore the triangle having vertices Y, I, A is a *golden triangle*, which is, by definition, an isosceles triangle with a ratio of side to base given by ϕ. This triangle decomposes into the two smaller triangles of vertices $\mathrm{Y}, \varepsilon\iota, \mathrm{A}$ and $\mathrm{A}, \varepsilon\iota, \mathrm{I}$, respectively. The first one is itself a golden triangle. Using the fundamental relation $\phi = 1 + \frac{1}{\phi}$ one can show that the second one is a *golden gnomon*, which is, by definition, an isosceles triangle with a ratio of side to base given by $\frac{1}{\phi}$ (see Figure 1). Now, if we consider the union of the smaller golden triangle with its reflection with respect to the Y I–axis, we obtain the thin rhombus, while to obtain the thick rhombus we consider the union of the golden gnomon with its reflection with respect to the AI–axis (see Figure 2).

The next step consists in outlining a correspondence between the Penrose rhombuses and some highly singular symplectic spaces. Some rather important results in symplectic geometry, namely the Atiyah, Guillemin-Sternberg convexity theorem [1, 7] and the Delzant classification theorem [6], allow to establish a correspondence between certain simple rational convex polygons and certain 4–dimensional compact symplectic manifolds that are symmetric with respect to a 2–dimensional torus. A similar correspondence can be found between suitable rational convex polytopes and higher-dimensional symplectic manifolds. For a thorough treatment of the theory of torus actions on compact symplectic manifolds we refer the reader to the book by M. Audin [2]. This correspondence has turned out to be very useful both for symplectic geometers, who use it to deduce properties of the manifolds from properties of the polygons, and for combinatorialists, who use it in the opposite direction, to deduce properties of the polygons from properties of the manifolds. Unfortunately, or maybe fortunately, this correspondence does not work for the Penrose rhombuses. In fact these polygons are not rational. However, a generalization of this correspondence by the author [9], allows to associate to any simple convex polygon (or, in higher-dimensions, any simple convex polytope), whether it is rational or not, a symplectic space that may have a very peculiar type of singularities. These spaces, which are called quasifolds, locally look like the quotient of an open subset of $\mathbb{R}^n$ ($\mathbb{R}^4$ in our current low dimensional situation) modulo the action of a discrete, possibly infinite group, and typically will not even be Hausdorff topological spaces. The group acting on the quasifold, is no longer a torus but is the quotient of a torus again by a discrete, possibly infinite group. This type of singularities is exactly what is needed in order to allow nonrationality of the corresponding polygons. As it turns out, the rhombuses correspond to quasifolds that are global quotients of the smooth symplectic manifold $S^2 \times S^2$ by infinite discrete groups. It is interesting to remark that the whole tiling gives rise only to two different such quotients, one corresponding to a thin rhombus tile, the other corresponding to a thick rhombus tile, and that these quotients are diffeomorphic.

References

[1] M. Atiyah, Convexity and Commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1–15.

[2] M. Audin, Torus Actions on Symplectic Manifolds, Birkhäuser (2004).

[3] F. Battaglia and E. Prato, The Symplectic Geometry of Penrose Rhombus tilings, in preparation.

[4] D. Brown, The Da Vinci code, Doubleday (2003).

[5] C.J. de Vogel, Pythagoras and Early Pythagoreanism, Van Gorcum (1966).

[6] T. Delzant, Hamiltoniennes Périodiques et Image Convexe de l'Application Moment, *Bull. S.M.F.* **116** (1988), 315–339.

[7] V. Guillemin and S. Sternberg, Convexity Properties of the Moment Mapping, *Invent. Math.* **67** (1982), 491–513.

[8] M. Livio, The Golden Ratio: The Story of Phi, the World's Most Astonishing Number, Broadway Books (2003).

[9] E. Prato, Simple Non-Rational Convex Polytopes via Symplectic Geometry, *Topology* **40** (2001), 961–975.

[10] E. Reiner and D. Pingree, Babylonian Planetary Omens. Part 1. The Venus Tablet of Ammisaduqa, Undena (1975).

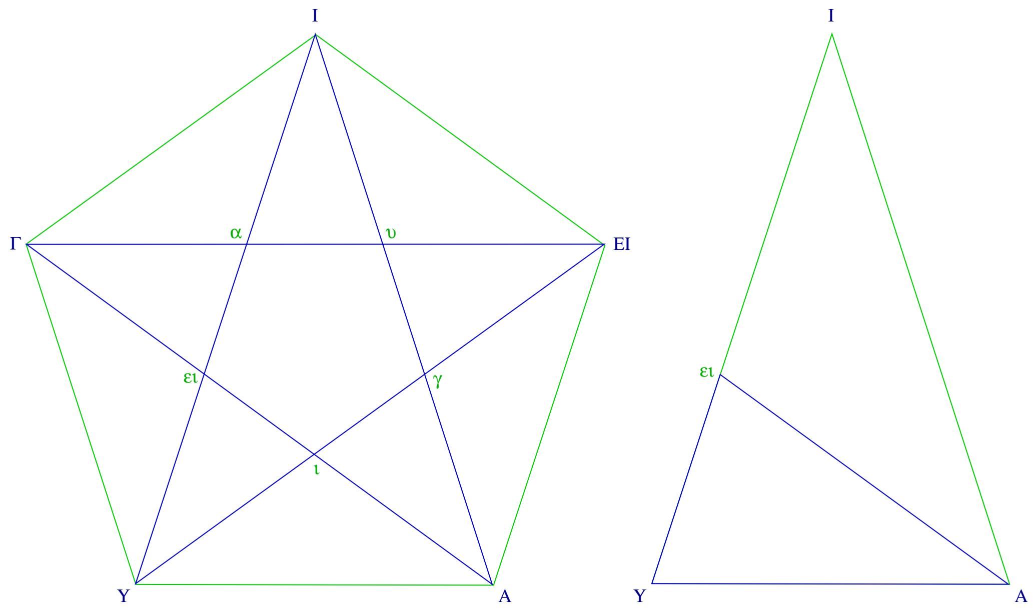

Figure 1: The pentagram inscribed in the pentagon and the decomposed golden triangle

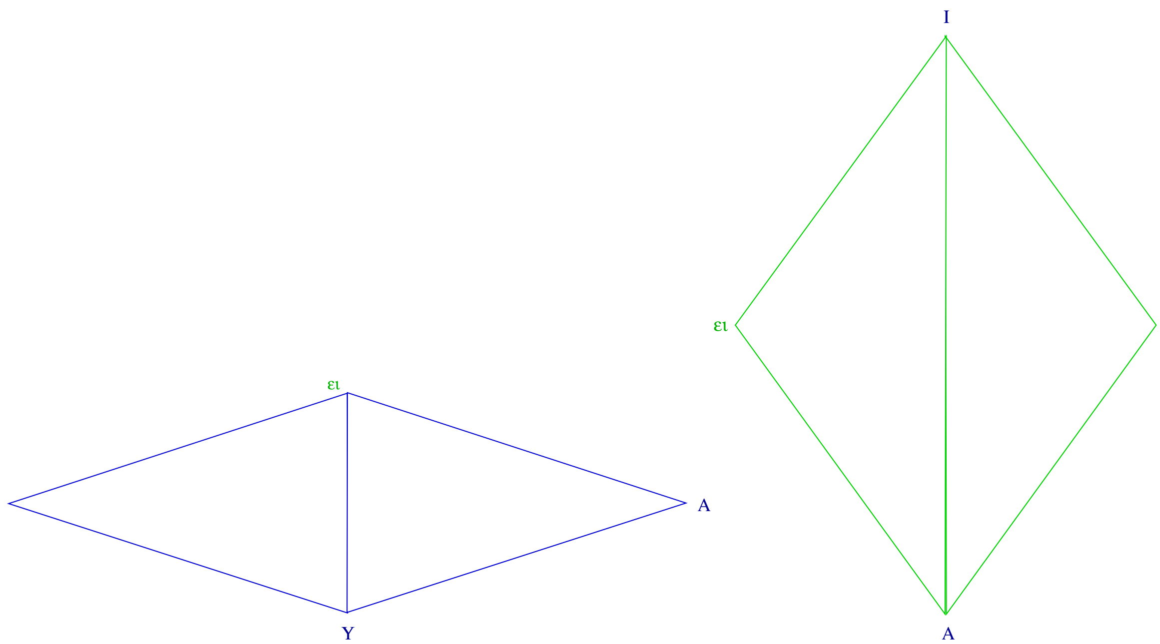

Figure 2: The thin rhombus and the thick rhombus