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Mathematical Connections
in Art, Music, and Science

Conveying Large Numbers to General Audiences

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Introduction

Almost daily we are presented in the media with important large numbers. For example, the number of dollars expended by the U.S. government on the cold war through the end of Reagan's administration was 10 trillion dollars. How much money is this? It is difficult for the average person to relate to such a large number. The late Carl Sagan had an answer to the question (which he originally posed in a speech delivered in 1988) [1]. His answer was to ask another question: What could you *buy* with 10 trillion dollars? The answer is startling: "Everything. You could buy everything in the United States, except the land. Everything."

Another example is the current world population figure of nearly 5.8 billion people, expected by the United Nations to grow to some 12 billion sometime in the middle of the next century (depending on the assumptions used). How big is this? If these people were placed side by side in a line, the line would stretch around the Earth 132 times!

The need to put large numbers into perspective is important for wise decisions affecting the well being of society and the health of democracy. The reason is that important judgements affecting public policy must be made by ordinary people and their elected representatives, judgements based on facts often presented using very large numbers, the sizes of which make it difficult for us to understand.

The purpose here is to discuss the problem of understanding, relating to, and possibly visualizing large numbers. Large numbers may be categorized as what I call pure numbers and those used to characterize real things. The pure numbers are not intended to represent any things in particular, or numbers of things. The largest known[1] prime number is an example of a pure number. It is a very big number, but it is probably of interest and direct relevance to few major policy decisions. Our attention here is directed exclusively toward the other kind of numbers, numbers with units or physical objects attached to them, such as dollars, people, years, stars in the sky, and distances, areas, and volumes.

Examples from the Literature

Authors, educators, public policy officials, and scientists have devised a number of interesting and often graphical ways to help their audiences understand the magnitudes of very large numbers. The examples given in the introduction are but two. Another clever one was provided by Harold Willens for the frontispiece of his book on the nuclear weapons crisis [2], shown in Fig. 1. He prints 121 small squares, in an 11 x 11 array. All but one are filled with 50 randomly arranged dots, each representing 3 megatons of conventional

[1] In 1953 Tobias Dantzig claimed the largest known prime number to be the *seventh Mersenne number*, calculated by the Institute of Numerical Analysis at Los Angeles to be 2^{2281} -1. The 1985 Guinness Book of World Records lists the largest prime number at that time as $2^{132,049}$ -1.

explosives–all the firepower of World War II. Three of the dots are circled, representing the 9 megatons of weaponry on just one Poseidon submarine, equal to the firepower of three World War II's, "and enough to destroy over 200 of the Soviet's largest cities. We have 31 such subs and 10 similar Polaris subs."

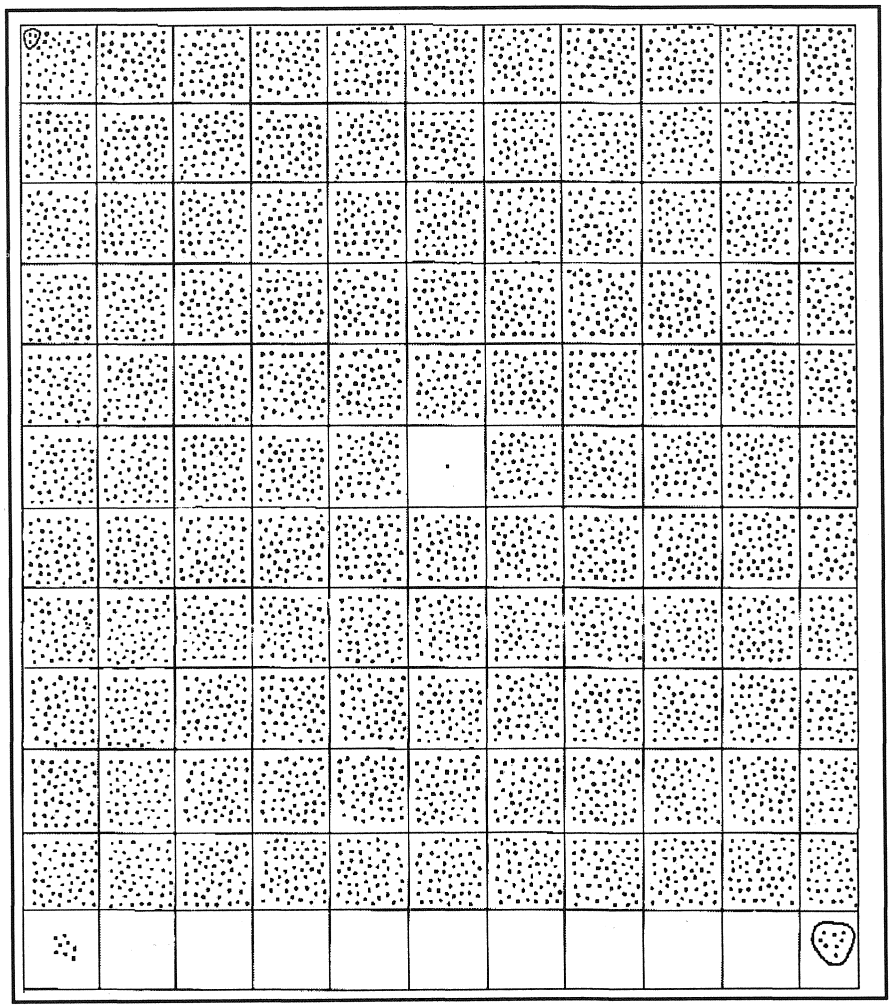

Figure 1. Chart of the explosive capacity of the world's nuclear weapon arsenal in 1984, according to Harold Willens in his book, *The Trimtab Factor*. Each dot represents all the firepower of World War II, 3 megatons. The total of all the dots is 18,000 megatons, or 6,000 World War II's. (The figure is truncated slightly on the right side.)

In an otherwise empty square below the filled ones are found 8 dots circled, representing the 24 megatons of nuclear explosives on just one Trident submarine, equal to the firepower of eight World War II's., "enough to destroy every major city in the northern hemisphere." Willens points out that at the time of his writing the Soviet Union had similar levels of destructive power. He goes on to say that just two squares on his chart (300 megatons) represent enough firepower to destroy all the large- and medium-sized cities in the entire world. The total firepower of all the 121 squares is 18,000 megatons, or 6,000 World War II's, equal to the world's total nuclear firepower in 1987.

I have seen and used presentations to audiences which attempt to help people understand large numbers. One is to let a small copper bb represent a reasonably sized fraction of the large number in discussion and to fill a container with enough of these to total the large number being illustrated. Bbs have the advantages of being small and relatively compact, and of making a loud noise when poured into a metal bowl or plate. One starts the demonstration by explaining what one bb represents. For example 1 million people. One bb is dropped into the metal bowl, making an audible clink. To give this number some perspective, a couple of cities having approximately 1 million people are mentioned, one from a developed country and one from an under-developed country. Then the population of New York city, for example, is dropped into the bowl, about 10 bbs, making a larger audible sound. Then the population of, say, Mexico City (20 bbs) is dropped into the bowl. Then the remaining 5,769 bbs, representing the remaining world population are poured slowly into the bowl, making a dramatic audible and visual effect.

The sound of bbs dropping is heard continuously for many seconds. As the bowl starts to fill, the clinking sound of bbs dropping into a hollow bowl is displaced by the lower, more thudding sound of bbs falling on top of bbs. When the bowl is filled, the audience can see that there are still a lot of bbs left in the source container. As these are poured in, the bowl overflows, and bbs drop to the floor, spreading helter skelter until, finally, all 5800 bbs representing world population have been poured. This gives a dramatic indication of how big 5.8 billion is.

Roy Beck uses a variation of this technique. In his video "Immigration by the Numbers," he uses differently colored gum balls, each one representing one million people [3]. He says that U.S. population increases by one million people, due to immigration each year. He says that some Americans are sincere in wanting to bring in large numbers of immigrants, to help alleviate third world suffering. His question is whether immigration is an effective tool to do this. Holding up one gum ball he says that "Each year, in our equanimity, we try to rescue this number from third world poverty. He drops one gum ball into a clear brandy snifter. Then he asks, "How many people are equally deserving?" He uses Mexico as a benchmark, pointing out that the average Mexican has a monetary income about one tenth of ours in the U.S. Then he asks how many people in the world are more impoverished than the average Mexican? The answer he gives is 4,600 million people, and he lifts up a large cylindrical jar from behind the podium, filled with 4,600 multicolored gum balls.

Beck says that these 4.6 billion people in the world are more impoverished than the average Mexican. If U.S. immigration policy is intended to help these people, he says, let's see how much the third world changes when we take one million of them each year out of poverty. He transfers one gum ball from the large jar into the much smaller brandy snifter and points out that the big jar has hardly changed its appearance at all, indicating that current U.S. immigration is a small drop in a large bucket of the problem. He points out that the 4,599 million people remaining in poverty in the third world are stuck there, saying that we have to figure out a way to help them in their own countries, not by letting them into the U.S.

Beck drops another gum ball into the U.S. brandy snifter, saying that "We can do this kind of thing forever but we won't make any difference in the world. There are many ways the U.S. can help the citizens of third world countries, but immigration is not one of them."

Then he brings up another issue—the U.S. as a safety valve for over-population in third world countries. "Last year we took in one million immigrants. Last year the third world added another eighty million people into the impoverished total." Then he pours 80 yellow gum balls into the big jar. "This year the U.S. will take in another million." One gum ball into the brandy snifter. "And the third world will add over 80 million more to the world." Eighty red gum balls go into the big jar. "Then another more than 80 million people are added to the world." 80 more gum balls into the big jar, causing it to overflow, with gum balls dropping to the floor and bouncing and rolling all over it. "There is no way we can ever get ahead of this, be a safety valve." This demonstration is a very effective way to illustrate large numbers in visual terms, mainly by comparison with smaller numbers that are easier to understand.

Let's turn our attention here to time. The Earth, for example, has been around for a very long time. Higher forms of life have existed for a relatively brief period, and human civilization has existed for just a blink of the eye in time, relative to the huge age of the Earth itself. One scheme devised for turning this immense span of time into something meaningful is to stretch out a long rope, 100 feet or so long, representing 100 million years of the Earth's existence, and to mark on this rope major events in history. The process is described by Dr. MagGregor Smith, Jr. in his wonderful book *Now That You Know* [4]. It is remarkable to see how short a period in the Earth's history we humans are using to extract most of the fossil fuels from the Earth.

This same approach, in the form of a graph, can be used to illustrate human population growth over a very long time period, showing the rapid, exponential-looking burst of population near the end of the graph. A similar chart is used to show the increase in energy consumption and other indicators of human impact on the Earth's resources.

The concept of exponential growth is a difficult one for many people to understand. It can lead to some very large numbers if the clock is allowed to tick long enough. In dealing with exponential growth, the doubling time is a very useful concept, as the following little story, told by Albert Bartlett, Prof. Emeritus of Physics at the University of Colorado, illustrates [5].

Legend has it that the game of chess was invented by a mathematician who worked for an ancient king. As a reward for the invention the mathematician asked for the amount of wheat that would be determined by the following process: He asked the king to place 1 grain of wheat on the first square of the chess board, double this and put 2 grains on the second square, and continue this way, putting on each square twice the number of grains that were on the preceding square. The filling of the chessboard is shown in the table.

Al's table of grains of wheat on the squares of a chessboard

Square number	Grains on square	Total grains thus far
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
64	2^{63}	$2^{64} - 1$

How much wheat is 2^{64} grains? Simple arithmetic shows that it is approximately 500 times the 1976 annual worldwide harvest of wheat! This amount is probably larger than all the wheat that has been harvested by humans in the history of the earth.

We see that on the last square one will place 2^{63} grains and the total number of grains on the board will then be one grain less than 2^{64}. How did we get to this enormous number? It is simple; we started with 1 grain of wheat and we doubled it a mere 63 times....

Speaking of large numbers, the 1985 edition of the Guinness Book of World Records says that the number 10^{100} is called a *Googol* and goes on to say that "ten raised to the power of a Googol is described as a Googolplex [6]. Some conception of the magnitude of such numbers can be gained when it is said that the number of electrons in some models of the observable universe probably does not exceed 10^{87}."

Human Perception of Magnitude

An interesting feature of the above examples is their use of *comparison* to illustrate the magnitude of large numbers. We humans are well-developed to see subtle differences in things, while absolute magnitudes can escape us. Thus, we best perceive the sizes of things by relating them to something else, to some standard of size with which we can compare them.

This is borne out in the literature on human perceptions of magnitude. Tobias Dantzig points out [7] that humans possess a faculty he calls *Number Sense,* an ability to see something added to or removed from a small collection of things. He surmises that counting things came later in our evolution. Some other species seem to have number sense, but not a counting ability. Number sense is inherently relative. "We enter a hall. Before us are two collections: the seats of the auditorium, and the audience. *Without counting* we can ascertain whether the two collections are equal and, if not equal, which is the greater. For if every seat is taken and no man is standing, *we know without counting* that there are more people than seats. We derive this knowledge through a process which dominates all mathematics and which has received the name of *one-*

to-one correspondence. It consists of assigning to every object of the collection an object of the other, the process being continued until one of the collections, or both, are exhausted."

Dantzig points out that "The limited scope of man's number sense makes it well nigh impossible to name a given number after some model-collection of which it is the 'cardinal measure'. The alternative is to associate the number with the symbols used in recording it.... With the advent of positional numeration and its universal acceptance, the decimal cryptogram of a number automatically provided it with a name. Today it has become more than a name: *we have learned to identify the number with its decimal cryptogram.* So great indeed is the force of habit, that most of us regard any other representation of a number as a sort of disguise; and this despite the fact that we all realize that there is nothing absolute or sacrosanct about the scale of ten."

The remarkable book *Powers of Ten* [8] deals with the need for understanding large things by their relationship to smaller ones in a very graphic way. The book starts with an image of distant stars 10^{25} meters away. Every two pages succeeding this are shown images and descriptions of objects on a scale that is ten times smaller than the preceding two pages. At 10^{11} meters we are in the middle of our solar system, at 10^{7} we are at the Earth, and at 10^{-3} m we are in a cell inside a human hand. The process continues down through atomic and subatomic levels to 10^{-16} m, inside the proton.

Powers of Ten illustrates well the problem we have in *really* understanding the meaning of large numbers—at magnitudes, large or small, well removed from our daily experiences. We can understand a change in size by *one* factor of ten reasonably well, but to understand a change in size that is *ten* factors of ten is a very difficult task.

Mathematical Literacy

One of the problems here is the mathematical illiteracy that permeates much of society. This is ironic. Modern humans have built and live within a very complex and technologically sophisticated society that depends upon a comprehensive knowledge of mathematics for its very functioning. And yet most of the members of that society are rank novices of both science and technology and the mathematics upon which they so strongly depend.

Perhaps one thing we are doing when we try to help an audience understand the meaning of large numbers is teach them a bit about mathematics, or at least numeration. If humans have an inherent number sense, as Dantzig seems to claim, the problem then is with the next step, taking them from the cardinals—where a number (or the size of something) is represented by a familiar image (or icon)— to the ordinals—an ordered sequence of characters representing increasing size. We have our fingers to use as devices for structuring a number system, but it is still very difficult to get much direct perception beyond a couple of factors of ten above the number of fingers on both hands.

If we plan to use our presentations about large numbers to teach elementary numeration concepts, perhaps we should turn to the literature on math education for guidance. One example is provided by Whitin, Mills, and O'Keefe [9] who argue that mathematics is a language: "It is a communication system that we use to explore and expand our knowledge of the world. Children grow in mathematical literacy when they have regular opportunities to investigate the purposes, processes, and content of the mathematical system." The approach used is to find something their math learners are interested in, such as a fascination with dinosaurs, and present them with opportunities to count dinosaurs and express their sizes. They describe a class in which children were encouraged to make up stories about dinosaurs. They were then asked to tell about any numbers that might have arisen in the telling of these stories. One student drew several dinosaurs, colored them, put them on "teams" differentiated by color, and included red spots to indicate battle wounds. "Thus Jermaine drew upon his knowledge of dinosaurs and integrated that learning through art, writing, mathematics, and language."

Developing more widespread mathematical literacy using such techniques, however noble, will not alone solve the problem addressed here, because even well-trained scientists and mathematicians have difficulty visualizing, perceiving, and really understanding the meaning of very large numbers. This is complicated by our inabilities to actually see with our eyes the whole thing. Who can see the entire population of the planet?

We end up trying to paint word pictures and draw analogies to large things we hope we can see and visualize better. How big is 5.8 billion humans? We turn to crippled artifacts and analogies, such as lining people up on a great circle around the Earth, showing how many times this line will circle the globe before we run out of humans, as if we could even comprehend how big the Earth is in the first place. Or we try to let a bb represent a single human. Since this produces too many of them for us to handle, we have to let each bb represent a million people. How realistic and understandable is this? If we have difficulty understanding how big a million people is, putting all of them into a tiny bb can't help much.

Our challenge is to find symbols to represent reasonably modest sized numbers that people can perceive fairly well, while not ending up with so many of these symbols that they won't fit on the page, in the container, or in the room. One million persons per bb is too many. We probably should first find a way to show how big a million people is. Then we can more effectively talk about a thousand million people (a billion). This is not an easy task. The reader is challenged to suggest new ways of visualizing or otherwise understanding very large numbers of things.

Bigness is relative. The key to presenting large numbers in understandable ways is to compare them to smaller things or processes which are of more direct experience. With really big numbers, this often has to be a multi-stage undertaking. Multi-stepped presentations, however, present difficulties. The steps need to be close together in time in order not to exceed attention-spans. If there are too many steps, the process can quickly become tedious and the main point of the presentation can be lost. I doubt if any more than two steps would be effective.

The Hidden Agenda

It is important to ask a critical question: What is the purpose of this exercise? Why is it so important for us to help people visualize large numbers? In most cases of presenting really large numbers to a general audience, there is a hidden agenda. The purpose is politics. We hope that by understanding the largeness of the numbers the audience will better understand the political problem being presented to them. It is hoped that they will see the problem as serious and will be motivated to join us in trying to solve it. Is it possible to separate this political motivation from the study of understanding large numbers? Maybe not, but it's worth a try.

For example, astronomers are accustomed to dealing with very large numbers. Teachers of astronomy face the problem of helping their students understand these large numbers. Seldom is this a political act. There may be some teachers who merely explain scientific notation and then use this to represent the large numbers they deal with, but most really want to help their students understand the meaning of the large numbers. The unit of length *light year* was devised by astronomers to make it easier to talk about large distances in the universe. The distance light travels in a year is a more convenient measure when talking about stars that are 60×10^{12} miles away (10 light years). It is kind of fun, when you are on a large college campus, to let some familiar object on campus represent our sun and then to place marks at various proportional distances from this "sun" around the campus to indicate the locations of the planets on an appropriately chosen scale.

I'm sure that there are additional areas where the provision of aids for understanding large numbers is not political. However, political motivation is predominant in most cases. In recent years we have seen politicians offering fairly sophisticated such aides: charts and graphs. In the 1996 election for President of the United States, candidate Ross Perot made use of graphs to explain concepts which seemed obvious to him (and, by implication, should be obvious to the average American). Unfortunately, linear graphs are not very helpful at understanding really large numbers, since the smaller numbers with which they are to be compared lie at the bottom of the graph, essentially indistinguishable from zero.

This is why logarithmic scales were developed. They are of little help to the non-mathematician, however, since our brains are accustomed to seeing our surroundings approximately linearly. (The major exception is our inherent understanding of the difference between the linear and angular size of an object, this distance lying at the heart of the phenomenon of perspective.) Even if Ross Perot switched to logarithmic scales for his graphs, I doubt if public understanding of the concepts would have increased much.

Conclusion

In attempting to find a graphical or symbolic scheme for representing large numbers, it is important first to recognize when the attempt is political in nature. This may aid in the choice of devices and symbols used in the presentation. Secondly, it is important to realize that what we are doing in most cases is to make the large numbers represented in an ordinal system clearer by expressing them cardinally, representing them symbolically in terms of collections of objects used to represent smaller collections of things. A consequence of this is that humans perceive sizes relatively. We should find devices in common experience, of inherently understandable magnitude, and use these as measures with which to gage the sizes of the large numbers being presented. This is not always easy. Whatever the basic metric is, it is likely so small that too many of them are needed to be practical.

This leads to the final conclusion, that whatever scheme is devised to make large numbers understandable will be the result of compromise. The symbols chosen will probably represent numbers themselves too large for full comprehension. The dramatic nature of the presentation has a lot to do with its impact on the audience. The latter can be both educational and political. It is fascinating to see how really big are the numbers we need to use in describing our world. This fascination can produce strong impact and drive home political or educational messages in the process.

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