# Supplementary Material for
# *The Looping Theorem in 2D and 3D Turtle Geometry*

Tom Verhoeff, Department of Mathematics and Computer Science, Eindhoven University of Technology, Netherlands; T.Verhoeff@tue.nl

## Examples for 2D Looping Theorem

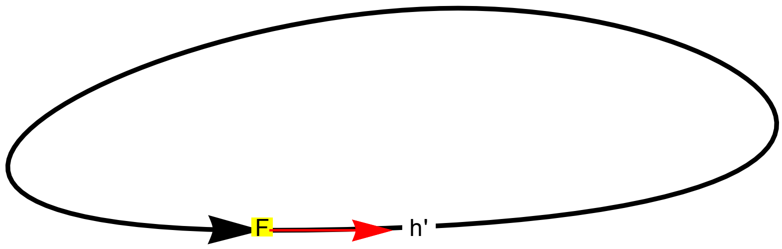


**Figure 1:** *$P$ is properly closed; $P^k$ overlays $P$ and hence is also properly closed.*

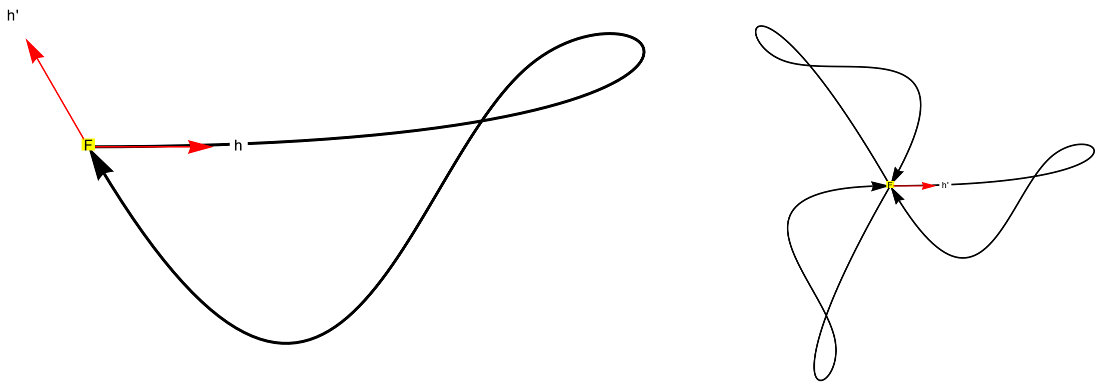


**Figure 2:** *$P$ (left) is closed and not proper ($\theta = 120°$); $P^k$ (right) is* properly *closed* $\Leftrightarrow$ $k \bmod 3 = 0$.

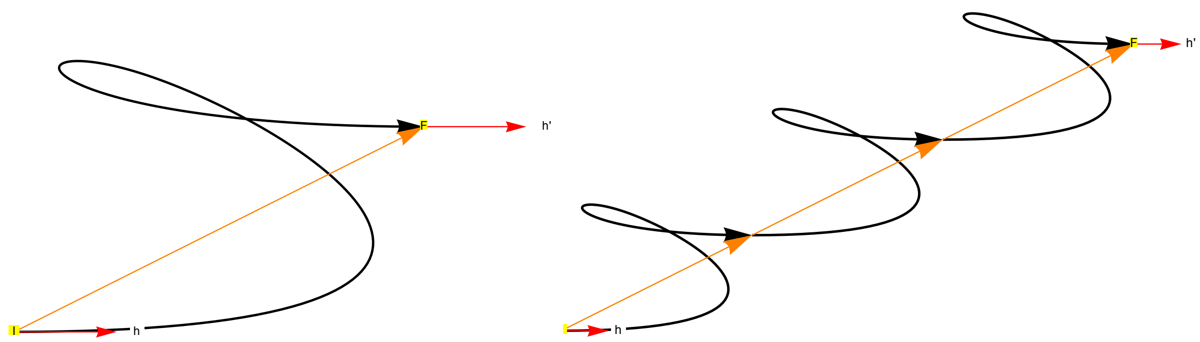


**Figure 3:** *$P$ (left) is proper ($\theta = 0°$) and not closed; $P^k$ (right) is proper and not closed.*

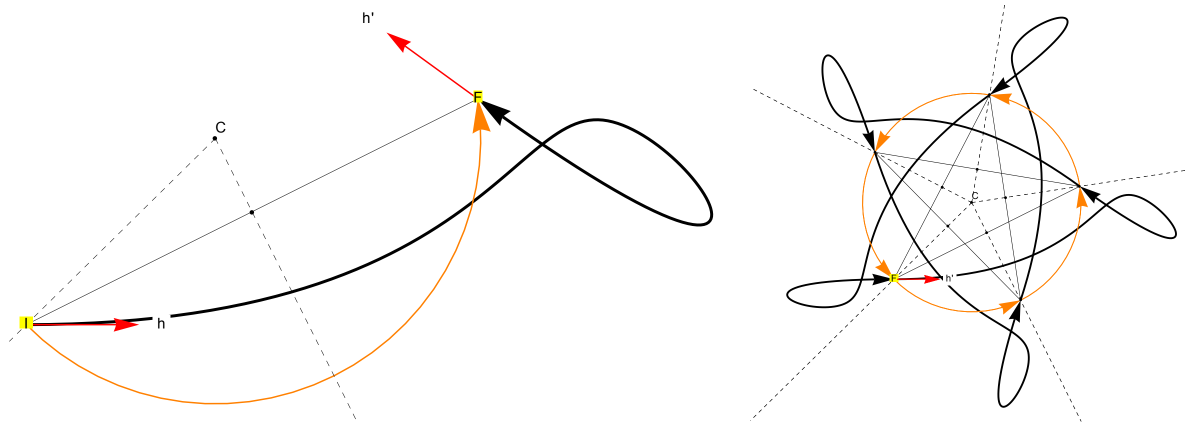


**Figure 4:** *P (left) is neither closed nor proper ($\theta$ = 144°); $P^k$ (right) is* properly closed $\Leftrightarrow$ $k$ mod 5 = 0.
*There is a separate movie that shows the initial state rotating to the final state.*

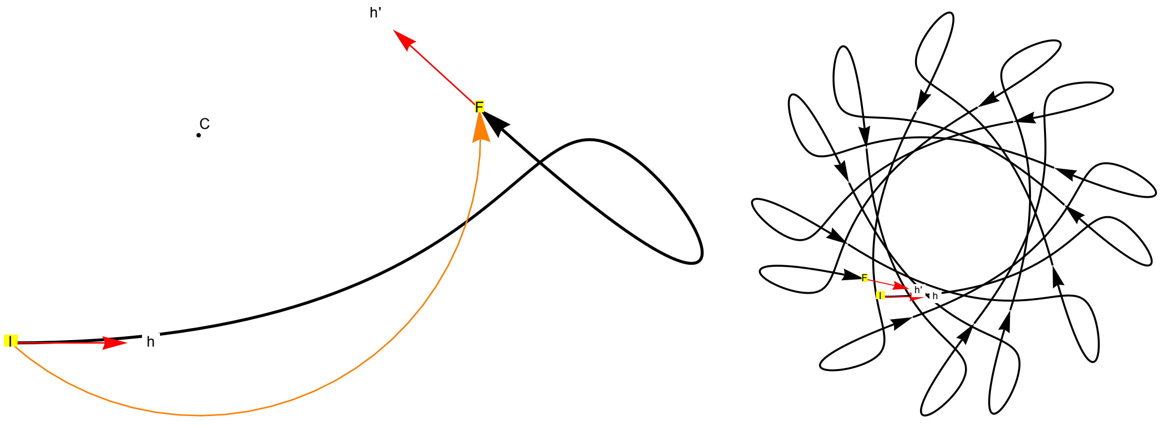


**Figure 5:** *P (left) is neither closed nor proper ($\theta$ = golden angle ≈ 137.5°);*
*$P^k$ (right) is neither closed nor proper.*

## Constructing the screw operation between two 3D turtle states

Given an initial and final turtle state in 3D, the screw operation that transforms the initial state to final state can be constructed with the following steps.

1. Translate (a copy of) the final attitude to the initial position.
2. Determine the Euler axis and Euler angle for the rotation that the transforms the initial state in that translated final state. There are many way to do so; see the survey [1]. A geometric construction is available in [2] (see Fig. 6).

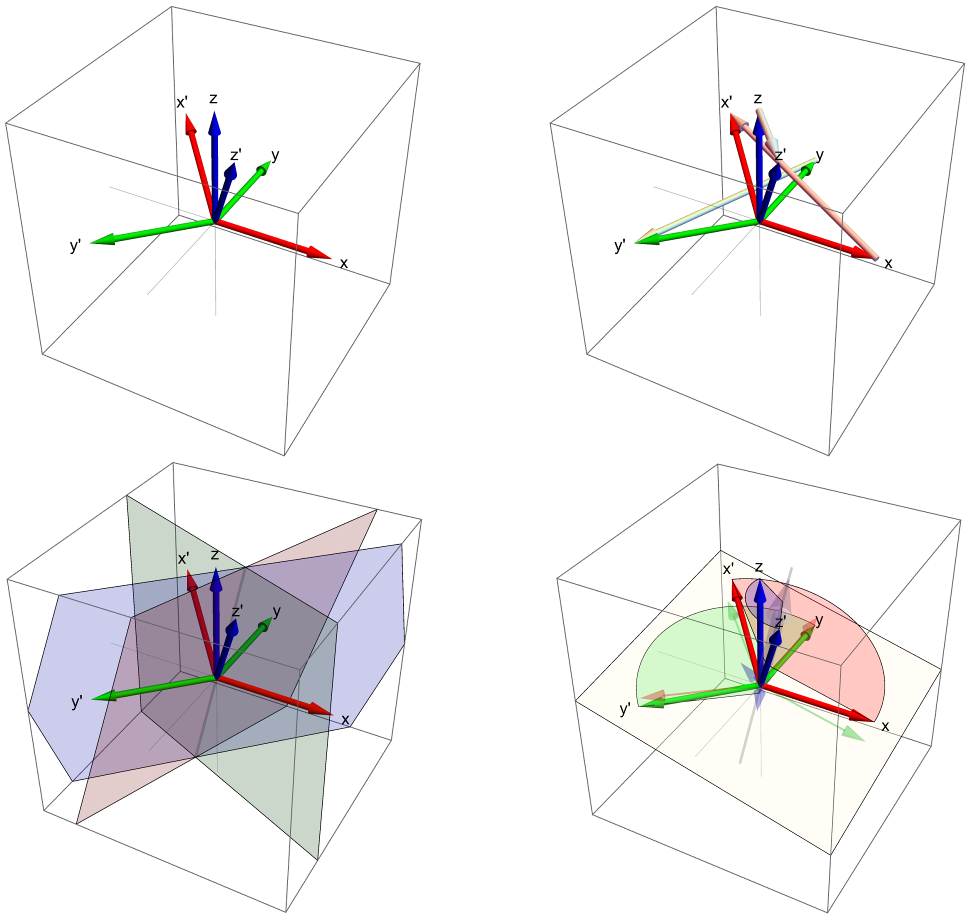


**Figure 6:** *Two turtle states $xyz$ and $x'y'z'$ (top left); pairwise displacements $x' - x$, $y' - y$, $z' - z$ (top right); bisector planes of displacements, intersecting in rotation axis (bottom left); projections on plane normal to axis with rotation angle (bottom right)*

3. The Euler axis gives the direction of the screw axis. (N.B. The Euler axis passes through the initial position, but the screw axis need not do that.)
4. The Euler angle is the angle $\theta$ of the screw operation.
5. Consider the plane with the Euler axis as normal vector, passing through the initial position. The distance between this plane and the final position is the translation distance $d$ of the screw operation.
6. Project the final position onto the plane of the preceding step, and find the rotation center for rotating the initial position to the projected final position over angle $\theta$. The screw axis passes through this rotation center. This part of the construction is the same as for the 2D case described in Fig. 1 of the paper.

## Examples for 3D Looping Theorem

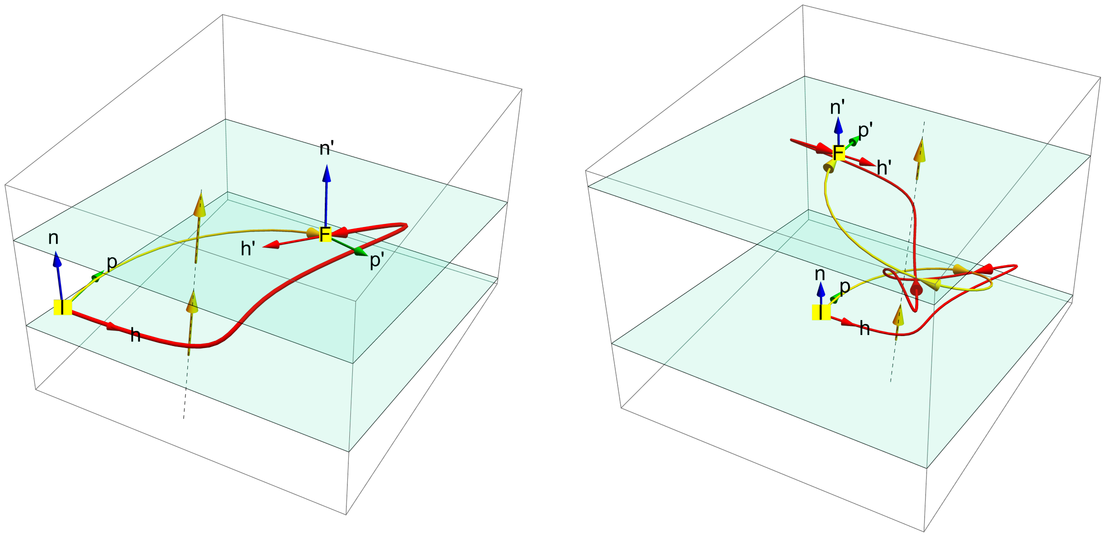


**Figure 7:** *P (left) is neither closed nor proper ($\theta = 120°$) and not planar ($d > 0$);*
*$P^k$ (right) is not closed, and it is proper $\Leftrightarrow k \bmod 3 = 0$.*

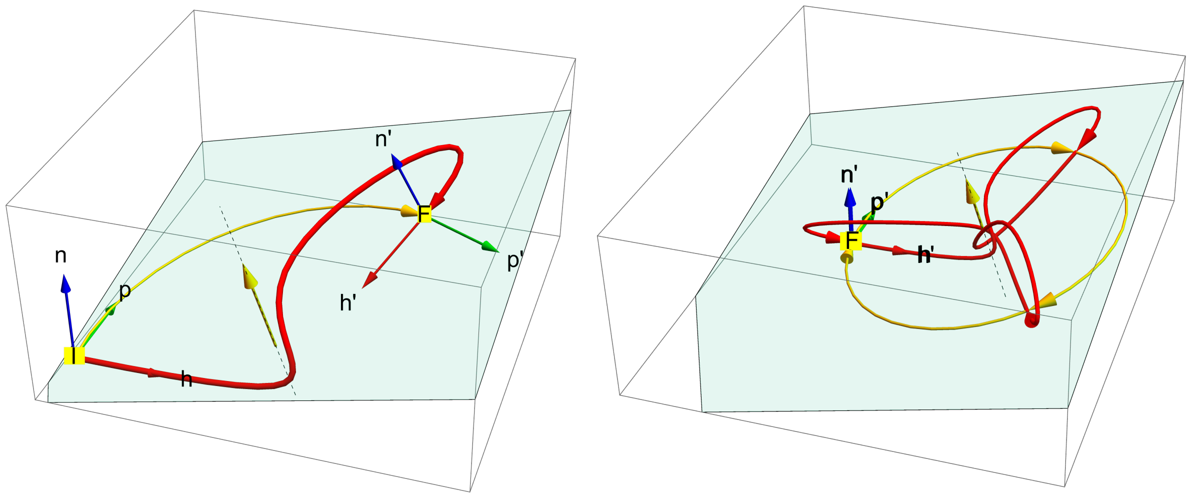


**Figure 8:** *P (left) is neither closed nor proper ($\theta = 120°$) but planar ($d = 0$);*
*$P^k$ (right) is closed, and proper $\Leftrightarrow k \bmod 3 = 0$;*
*$P^3$ is knotted, not self-intersecting with winding number 2.*

## References


[1] S. Sarabandi and F. Thomas. "A Survey on the Computation of Quaternions From Rotation Matrices." *Journal of Mechanisms and Robotics*, vol. 11, no. 2, 03 2019, 021006. https://doi.org/10.1115/1.4041889.

[2] T. Verhoeff. "Compute the Euler Axis and Angle Geometrically." Wolfram Demonstrations Project. May 2021. http://demonstrations.wolfram.com/ComputeTheEulerAxisAndAngleGeometrically/.