NOTES ON KONIGSBERG BRIDGES EULERIZATIONS
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The following pages contain Euler paths generated using *Mathematica* sufficient to count all possible distinct Euler paths on a labeled version of the Königsberg bridges graph in which one edge is allowed to be traversed twice. By "labeled" we mean that vertices and edges are distinguished, for example by verbs or nouns – or for the denizens of 1700s Konigsberg, by the fact that bridges and land masses are distinct from each other! For example, ancient Königsberg Giblets Bridge is certainly different from its Blacksmith Bridge!

Here, by the way, is what Euler wrote about his understanding of the complexity of tabulating the actual numbers of possible paths: "The particular problem of the seven bridges of Konigsberg could be solved by carefully tabulating all possible paths, thereby ascertaining by inspection which of them, if any, met the requirement. This method of solution, however, is too tedious and too difficult because of the large number of possible combinations, and in other problems where many more bridges are involved it could not be used at all."

- From a translation of Euler's original paper: "The Seven Bridges of Konigsberg." *The World of Mathematics*, Vol 1, pg 574, James R. Newman, 1956, Simon and Schuster.

For ease of computation, in the following pages I've labeled the vertices as 1 to 4 and the edges as 1 to 7. In all diagrams, one edge is the "doubled" edge which those attempting to walk the bridges would experience as the bridge they traverse twice. Note that even though bridges 1 and 2, for example, connect the same land masses, they are different bridges.

<u>Symmetry considerations</u>: for example, for the paths starting at vertex 1 and ending at vertex 3 in the graph in which bridge 4 is the doubled edge, 12 Euler paths begin with edge 1 and end with edge 6. Then also 12 will begin with edge 1 and end with edge 7, and by symmetry the same numbers will begin with edge 2 and end with either edge 6 or 7. Finally, we can reverse all these paths, doubling the total numbers. Such symmetry factors simplify the counting. This *Mathematica* routine essentially throws out the non-working paths from the list of 8!/2 = 20,160 – not a great algorithm for a slightly larger graph, since factorials grow faster than exponentials!

Here is the *Mathematica* code I wrote to tabulate the Euler paths. We need to use the symmetries to count all the paths, and also double all numbers since every path may be reversed to give a distinct path.

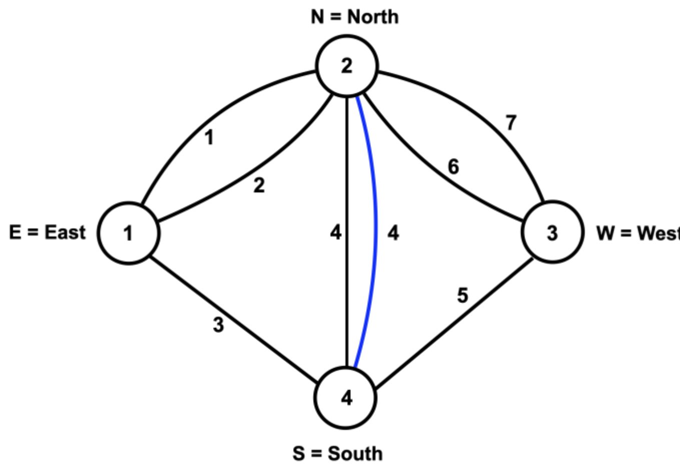
```
nextEdges = {{1, 2, 3}, {1, 2, 4, 6, 7}, {5, 6, 7}, {3, 4, 5}};
nextVertex[{1, 1}] = 2; nextVertex[{1, 2}] = 2; nextVertex[{1, 3}] = 4;
nextVertex[{2, 1}] = 1; nextVertex[{2, 2}] = 1; nextVertex[{2, 4}] = 4;
nextVertex[{2, 6}] = 3; nextVertex[{2, 7}] = 3;
nextVertex[{3, 5}] = 4; nextVertex[{3, 6}] = 2; nextVertex[{3, 7}] = 2;
nextVertex[{4, 3}] = 1; nextVertex[{4, 4}] = 2; nextVertex[{4, 5}] = 3;
edgePerms = Permutations[{1, 2, 3, 4, 5, 5, 6, 7}];
edgePermsLength = Length[edgePerms];

startVertex = 1; endVertex = 2; startEdge = 3; endEdge = 6;
edgesVsAndEs = {};
Do[If[(edgePerms[[i]][[1]] == startEdge) && (edgePerms[[i]][[8]] == endEdge),
   edgesVsAndEs = Join[edgesVsAndEs, {edgePerms[[i]]}]], {i, 1, edgePermsLength}];
edgesVsAndEsLength = Length[edgesVsAndEs];

eulerVsandEs = {};
Do[j = 1; eulertest = True; currentVx = startVertex;
  While[j ≤ 7,
   If[MemberQ[nextEdges[[currentVx]], edgesVsAndEs[[i]][[j]]],
    currentVx = nextVertex[{currentVx, edgesVsAndEs[[i]][[j]]}];
    j++, j = 8;
    eulertest = False]];
  If[eulertest == True, eulerVsandEs = Join[eulerVsandEs, {edgesVsAndEs[[i]]}]],
  {i, 1, edgesVsAndEsLength}];
[221]:=
Print["StartVx = ", startVertex, ", EndVx = ", endVertex, ", StartEdge = ", startEdge, ", EndEdge = ",
  endEdge, ", # Euler Paths = ", Length[eulerVsandEs]];
eulerVsandEs
```

The next page shows the number appears to be somewhat surprisingly high for this small graph (if the calculations are correct!): 1296 of the 8!/2= 20,160 permutations of the numbers 1 to 7, with one number repeated, represent the distinct Euler paths in which one edge is traversed twice.

Euler Paths over the Konigsberg Bridges

Doubled edge	Start vertex	End vertex	Start edge	End edge	# paths	symmetric factor	Reversal factor	Total
4	1	3	1	6	12	4	2	96
4	1	3	1	5	12	2	2	48
4	1	3	3	6	12	2	2	48
4	1	3	3	5	8	1	2	16
							Subtotal	208
5	1	2	1	2	8	4	2	64
5	1	2	1	4	8	4	2	64
5	1	2	1	6	8	4	2	64
5	1	2	3	1	8	4	2	64
5	1	2	3	4	4	2	2	16
5	1	2	3	6	6	4	2	48
							Subtotal	320
1	3	4	5	4	12	4	2	96
1	3	4	5	3	12	4	2	96
1	3	4	6	3	12	8	2	192
1	3	4	6	4	12	8	2	192
1	3	4	6	5	12	8	2	192
							Subtotal	768
							TOTAL	**1296**

Symmetry factor note: for example, for starting vertex 3 and ending vertex 4, the pattern will repeat if starting edge is 7 instead of 6.
The pattern will also repeat if the doubled edge is either 2, 6, or 7 instead of 1. This all results in a symmetry factor of 8.



StartVx = 1, EndVx = 3, StartEdge = 3, EndEdge = 5, ♯ Euler Paths = 8

ut[126]=

{{3, 4, 1, 2, 6, 7, 4, 5}, {3, 4, 1, 2, 7, 6, 4, 5}, {3, 4, 2, 1, 6, 7, 4, 5}, {3, 4, 2, 1, 7, 6, 4, 5},
{3, 4, 6, 7, 1, 2, 4, 5}, {3, 4, 6, 7, 2, 1, 4, 5}, {3, 4, 7, 6, 1, 2, 4, 5}, {3, 4, 7, 6, 2, 1, 4, 5}}

StartVx = 1, EndVx = 3, StartEdge = 3, EndEdge = 6, ♯ Euler Paths = 12

ut[110]=

{{3, 4, 1, 2, 4, 5, 7, 6}, {3, 4, 1, 2, 7, 5, 4, 6}, {3, 4, 2, 1, 4, 5, 7, 6}, {3, 4, 2, 1, 7, 5, 4, 6},
{3, 4, 4, 5, 7, 1, 2, 6}, {3, 4, 4, 5, 7, 2, 1, 6}, {3, 4, 7, 5, 4, 1, 2, 6}, {3, 4, 7, 5, 4, 2, 1, 6},
{3, 5, 7, 1, 2, 4, 4, 6}, {3, 5, 7, 2, 1, 4, 4, 6}, {3, 5, 7, 4, 4, 1, 2, 6}, {3, 5, 7, 4, 4, 2, 1, 6}}

StartVx = 1, EndVx = 3, StartEdge = 1, EndEdge = 5, ♯ Euler Paths = 12

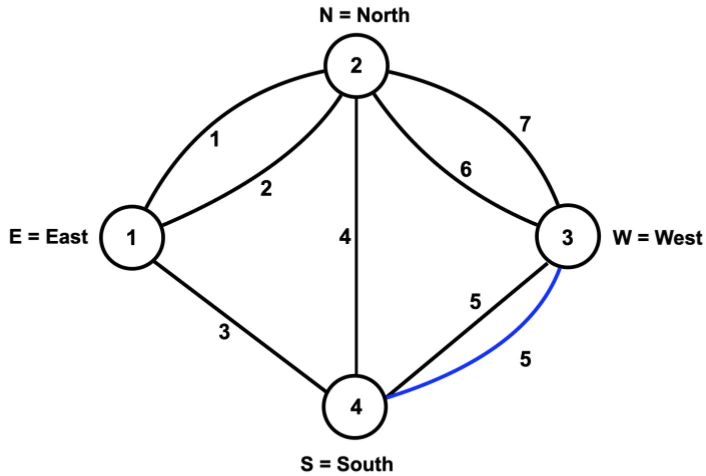
ut[94]=

{{1, 2, 3, 4, 6, 7, 4, 5}, {1, 2, 3, 4, 7, 6, 4, 5}, {1, 4, 3, 2, 6, 7, 4, 5}, {1, 4, 3, 2, 7, 6, 4, 5},
{1, 4, 4, 6, 7, 2, 3, 5}, {1, 4, 4, 7, 6, 2, 3, 5}, {1, 6, 7, 2, 3, 4, 4, 5}, {1, 6, 7, 4, 3, 2, 4, 5},
{1, 6, 7, 4, 4, 2, 3, 5}, {1, 7, 6, 2, 3, 4, 4, 5}, {1, 7, 6, 4, 3, 2, 4, 5}, {1, 7, 6, 4, 4, 2, 3, 5}}

StartVx = 1, EndVx = 3, StartEdge = 1, EndEdge = 6, ♯ Euler Paths = 12

ut[78]=

{{1, 2, 3, 4, 4, 5, 7, 6}, {1, 2, 3, 4, 7, 5, 4, 6}, {1, 2, 3, 5, 7, 4, 4, 6}, {1, 4, 3, 2, 4, 5, 7, 6},
{1, 4, 3, 2, 7, 5, 4, 6}, {1, 4, 4, 2, 3, 5, 7, 6}, {1, 4, 4, 7, 5, 3, 2, 6}, {1, 4, 5, 7, 2, 3, 4, 6},
{1, 4, 5, 7, 4, 3, 2, 6}, {1, 7, 5, 3, 2, 4, 4, 6}, {1, 7, 5, 4, 2, 3, 4, 6}, {1, 7, 5, 4, 4, 3, 2, 6}}



StartVx = 1, EndVx = 2, StartEdge = 3, EndEdge = 6, ♯ Euler Paths = 6

ut[222]=

{{3, 4, 1, 2, 7, 5, 5, 6}, {3, 4, 2, 1, 7, 5, 5, 6}, {3, 5, 5, 4, 1, 2, 7, 6},
{3, 5, 5, 4, 2, 1, 7, 6}, {3, 5, 7, 1, 2, 4, 5, 6}, {3, 5, 7, 2, 1, 4, 5, 6}}

StartVx = 1, EndVx = 2, StartEdge = 3, EndEdge = 4, ♯ Euler Paths = 4

ut[206]=

{{3, 5, 6, 1, 2, 7, 5, 4}, {3, 5, 6, 2, 1, 7, 5, 4}, {3, 5, 7, 1, 2, 6, 5, 4}, {3, 5, 7, 2, 1, 6, 5, 4}}

StartVx = 1, EndVx = 2, StartEdge = 3, EndEdge = 1, ♯ Euler Paths = 8

ut[190]=

{{3, 4, 6, 5, 5, 7, 2, 1}, {3, 4, 7, 5, 5, 6, 2, 1}, {3, 5, 5, 4, 6, 7, 2, 1}, {3, 5, 5, 4, 7, 6, 2, 1},
{3, 5, 6, 4, 5, 7, 2, 1}, {3, 5, 6, 7, 5, 4, 2, 1}, {3, 5, 7, 4, 5, 6, 2, 1}, {3, 5, 7, 6, 5, 4, 2, 1}}

StartVx = 1, EndVx = 2, StartEdge = 1, EndEdge = 6, ♯ Euler Paths = 8

ut[174]=

{{1, 2, 3, 4, 7, 5, 5, 6}, {1, 2, 3, 5, 5, 4, 7, 6}, {1, 2, 3, 5, 7, 4, 5, 6}, {1, 4, 3, 2, 7, 5, 5, 6},
{1, 4, 5, 5, 3, 2, 7, 6}, {1, 4, 5, 7, 2, 3, 5, 6}, {1, 7, 5, 3, 2, 4, 5, 6}, {1, 7, 5, 4, 2, 3, 5, 6}}

StartVx = 1, EndVx = 2, StartEdge = 1, EndEdge = 4, ♯ Euler Paths = 8

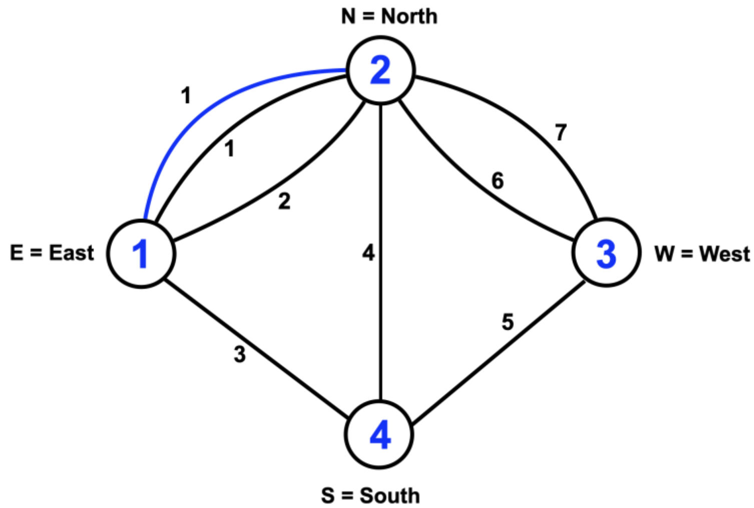
ut[158]=

{{1, 2, 3, 5, 6, 7, 5, 4}, {1, 2, 3, 5, 7, 6, 5, 4}, {1, 6, 5, 3, 2, 7, 5, 4}, {1, 6, 5, 5, 7, 2, 3, 4},
{1, 6, 7, 2, 3, 5, 5, 4}, {1, 7, 5, 3, 2, 6, 5, 4}, {1, 7, 5, 5, 6, 2, 3, 4}, {1, 7, 6, 2, 3, 5, 5, 4}}

StartVx = 1, EndVx = 2, StartEdge = 1, EndEdge = 2, ♯ Euler Paths = 8

ut[142]=

{{1, 4, 5, 6, 7, 5, 3, 2}, {1, 4, 5, 7, 6, 5, 3, 2}, {1, 6, 5, 4, 7, 5, 3, 2}, {1, 6, 5, 5, 7, 4, 3, 2},
{1, 6, 7, 4, 5, 5, 3, 2}, {1, 7, 5, 4, 6, 5, 3, 2}, {1, 7, 5, 5, 6, 4, 3, 2}, {1, 7, 6, 4, 5, 5, 3, 2}}



StartVx = 1, EndVx = 2, StartEdge = 3, EndEdge = 6, # Euler Paths = 6

Out[222]=
{{3, 4, 1, 2, 7, 5, 5, 6}, {3, 4, 2, 1, 7, 5, 5, 6}, {3, 5, 5, 4, 1, 2, 7, 6},
{3, 5, 5, 4, 2, 1, 7, 6}, {3, 5, 7, 1, 2, 4, 5, 6}, {3, 5, 7, 2, 1, 4, 5, 6}}

StartVx = 1, EndVx = 2, StartEdge = 3, EndEdge = 4, # Euler Paths = 4

Out[206]=
{{3, 5, 6, 1, 2, 7, 5, 4}, {3, 5, 6, 2, 1, 7, 5, 4}, {3, 5, 7, 1, 2, 6, 5, 4}, {3, 5, 7, 2, 1, 6, 5, 4}}

StartVx = 1, EndVx = 2, StartEdge = 3, EndEdge = 1, # Euler Paths = 8

Out[190]=
{{3, 4, 6, 5, 5, 7, 2, 1}, {3, 4, 7, 5, 5, 6, 2, 1}, {3, 5, 5, 4, 6, 7, 2, 1}, {3, 5, 5, 4, 7, 6, 2, 1},
{3, 5, 6, 4, 5, 7, 2, 1}, {3, 5, 6, 7, 5, 4, 2, 1}, {3, 5, 7, 4, 5, 6, 2, 1}, {3, 5, 7, 6, 5, 4, 2, 1}}

StartVx = 1, EndVx = 2, StartEdge = 1, EndEdge = 6, # Euler Paths = 8

Out[174]=
{{1, 2, 3, 4, 7, 5, 5, 6}, {1, 2, 3, 5, 5, 4, 7, 6}, {1, 2, 3, 5, 7, 4, 5, 6}, {1, 4, 3, 2, 7, 5, 5, 6},
{1, 4, 5, 5, 3, 2, 7, 6}, {1, 4, 5, 7, 2, 3, 5, 6}, {1, 7, 5, 3, 2, 4, 5, 6}, {1, 7, 5, 4, 2, 3, 5, 6}}

StartVx = 1, EndVx = 2, StartEdge = 1, EndEdge = 4, # Euler Paths = 8

Out[158]=
{{1, 2, 3, 5, 6, 7, 5, 4}, {1, 2, 3, 5, 7, 6, 5, 4}, {1, 6, 5, 3, 2, 7, 5, 4}, {1, 6, 5, 5, 7, 2, 3, 4},
{1, 6, 7, 2, 3, 5, 5, 4}, {1, 7, 5, 3, 2, 6, 5, 4}, {1, 7, 5, 5, 6, 2, 3, 4}, {1, 7, 6, 2, 3, 5, 5, 4}}

StartVx = 1, EndVx = 2, StartEdge = 1, EndEdge = 2, # Euler Paths = 8

Out[142]=
{{1, 4, 5, 6, 7, 5, 3, 2}, {1, 4, 5, 7, 6, 5, 3, 2}, {1, 6, 5, 4, 7, 5, 3, 2}, {1, 6, 5, 5, 7, 4, 3, 2},
{1, 6, 7, 4, 5, 5, 3, 2}, {1, 7, 5, 4, 6, 5, 3, 2}, {1, 7, 5, 5, 6, 4, 3, 2}, {1, 7, 6, 4, 5, 5, 3, 2}}