Polyhedral-Edge Knots

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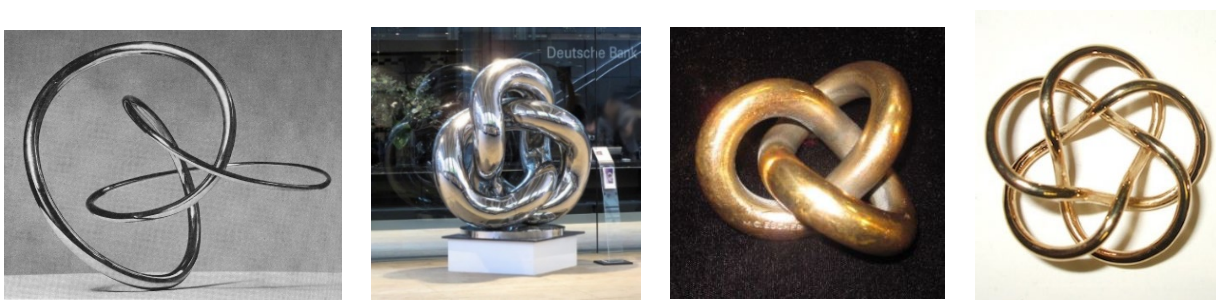
Abstract

Mathematical knots are severely restricted in the symmetries they can exhibit; they cannot assume the symmetries of the Platonic solids. An approach is presented that forms tubular knot sculptures that display in their overall structure the shapes of regular or semi-regular polyhedra: The knot strand passes along all the edges of such a polyhedron once or twice, so that an Eulerian circuit can be formed. Results are presented in the form of small 3D prints.

1. Introduction

Several artists have used tubular representations of mathematical knots and links to make attractive constructivist sculptures (Figure 1). Often, they try to make these sculptures as symmetrical as possible (Figures 1c, 1d); but there are restrictions how symmetrical a mathematical knot can be made. Section 2 reviews the symmetry types that are achievable by mathematical knots. Subsequently I explore ways in which I can design knot sculptures so that their overall shape approximates the higher-order symmetries exhibited by the Platonic and Archimedean solids.



(a) (b) (c) (d)

Figure 1: *Knot sculptures: (a) de Rivera: "Construction #35" [2]. (b) Zawitz: "Infinite Trifoil" [11]. (c) Escher: "The Knot" [3]. (d) Finegold: "Torus (3,5) Knot" [4].*

2. The Symmetries of Knots

All finite 3-dimensional objects fall into one of 14 possible symmetry families. First, there are the seven roughly spherical *Platonic symmetries* (Figure 2) derived from the regular and semi-regular polyhedra. The first three are: $\mathbf{T_d}$, the tetrahedral symmetry; $\mathbf{O_h}$, the octahedral (or cube) symmetry; and $\mathbf{I_h}$, the icosahedral (or dodecahedral) symmetry. If we suppress all mirror symmetries by placing chiral markings on all the faces, we obtain the oriented versions **T**, **O**, and **I**, of these symmetries. Finally, there is $\mathbf{T_h}$ the symmetry of the *oriented double-tetrahedron*, the aligned combination of two tetrahedra with opposite orientation.

The next seven families are the *prismatic symmetries*. They all have a dominant *n*-fold rotational symmetry axis, where *n* can range from 1 to infinity. They are distinguished by what additional symmetry operations they allow. The various possibilities are most easily understood by starting with the seven 2-dimensional frieze symmetries (Figure 3). A sequence of *n* cells of such a frieze is then wrapped around a cylindrical or prismatic body to produce *n*-fold rotational symmetry around the dominant axis.

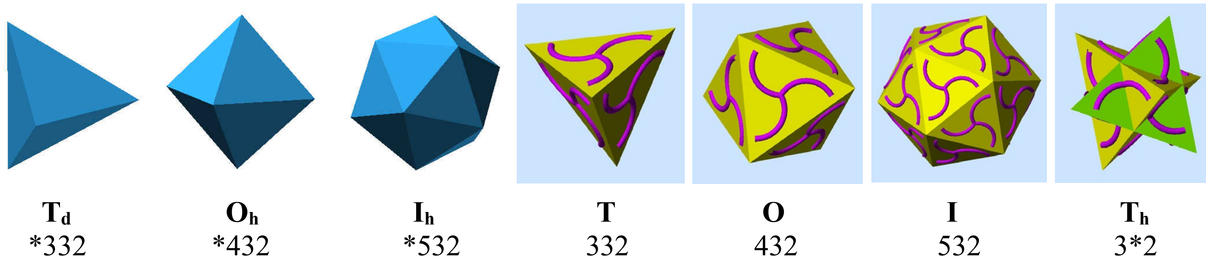


Figure 2: *The Platonic symmetries: (top) Schönflies notation; (bottom) Conway's orbifold notation.*

If there are no other symmetry operations possible beyond the rotations around the dominant axis, we have the *cyclic family* $\mathbf{C_n}$ (Figure 3a, top). To this rotational symmetry, we may add different mirroring operations. If we add a mirror operation along the dominant axis, we obtain family $\mathbf{C_{nh}}$. If instead we add *n* mirror planes containing the dominant axis, we obtain family $\mathbf{C_{nv}}$. Finally, there is a way of combining rotation and a mirror operation to produce *glide symmetry*. If we can rotate a shape through 180°/*n* degrees and apply mirroring along the dominant axis to bring the shape into coincidence with itself, we have the family $\mathbf{S_{2n}}$.

If, in addition to the rotational symmetry around the dominant axis, there are *n* 2-fold rotation axes perpendicular to the dominant rotation axes (shown by blue dots in Figure 3b), we are facing a member of the *dihedral family*. If we start with the basic dihedral symmetry $\mathbf{D_n}$ (Figure 3b, top) and add to it either the horizontal or the vertical mirror planes, as we did for the $\mathbf{C_n}$ family, we obtain $\mathbf{D_{nh}}$ symmetry, which contains *both* these mirroring operations; this happens because of the *n* 2-fold rotation axes. But there is also a dihedral family that exhibits glide symmetry, this is called $\mathbf{D_{nd}}$. Through this paper, I will use the Schönflies notation for the various symmetries. There are several other symmetry notations. For a broader perspective, I have also included Conway's orbifold notation in Figures 2 and 3.

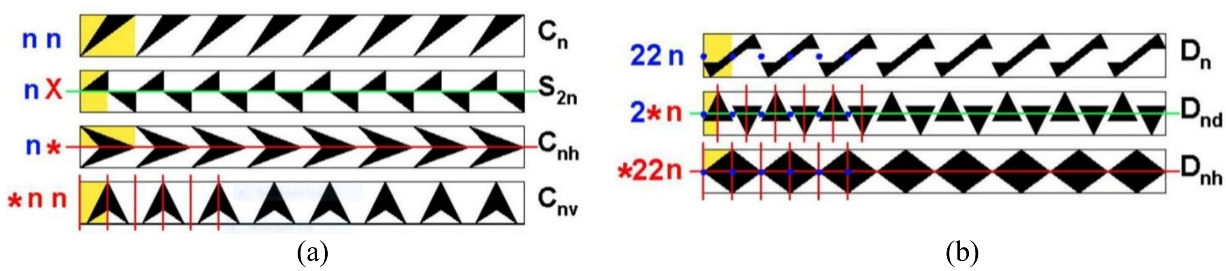
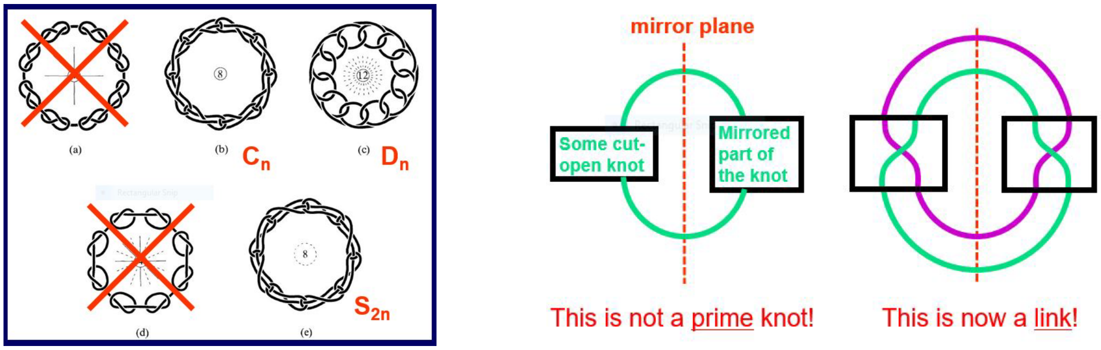


Figure 3: *Frieze / prismatic) symmetries: (a) the 4 cyclic families, (b) the 3 dihedral families.*

When trying to design tubular sculptures representing mathematical knots with as much symmetry as possible, one runs into some fundamental limitations concerning the symmetries that knots can exhibit. Because mathematical knots must consist of a single closed space-loop that must not self-intersect, they can only fit into five of the prismatic families (Figure 4a), as Grünbaum et al. have shown. The proof starts with a demonstration that no knot can exhibit more than one rotational symmetry axis with a valence higher than two [5].

If we constrain ourselves to consider only non-trivial *prime knots*, then they will all belong to just the three prismatic families: $\mathbf{C_n}$, $\mathbf{D_n}$, and $\mathbf{S_{2n}}$. Thus, specifically, prime knots cannot exhibit mirror symmetry. Figures 4b and 4c illustrate this point. Figure 4b shows an attempted construction of a mirror-symmetric knot. But with just two edges connecting the two mirrored parts, this constitutes a *compound knot*. To turn this into a prime knot, at least one more pair of connecting edges needs to be added. If one tries to do this while keeping the overall figure mirror symmetric, one always obtains a *multi-component link* rather than a single mathematical prime knot (Figure 4c).



(a) (b) (c)

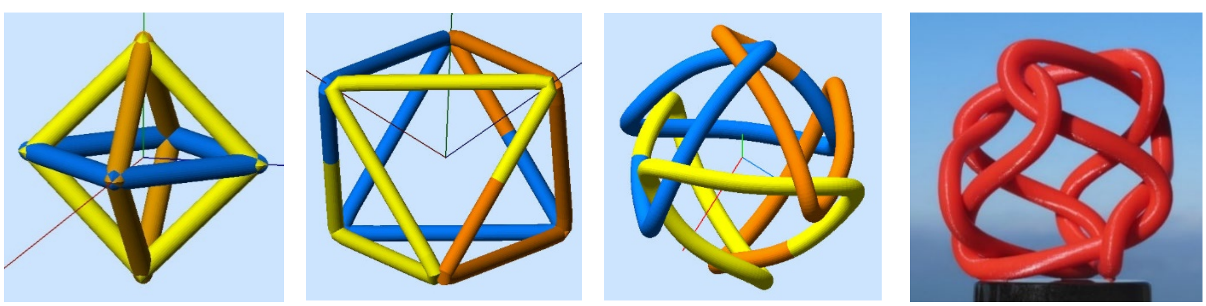
Figure 4: *Limitations of knot symmetries: (a) as discussed by Grünbaum [5]. (b,c) Exclusion of mirror symmetry in non-trivial prime knots.*

3. Knots Based on (Semi-)Regular Polyhedra

To make knot sculptures that appear to be highly symmetrical, let's step away from the approach of directly constructing a knot based on one of the three allowed prismatic frieze groups. Instead, let's look at a process that starts with the Platonic or Archimedean solids – even though we know that the final knot cannot possibly have the full symmetry of these polyhedra. My aim is to make sculptures, where the overall structure is inspired by a wire-frame representation of a Platonic solid, and where the knot strand runs along all the edges of such a regular polyhedron. This creates "polyhedral-edge" or "wire-frame" knot sculptures.

Edge Knot Based on the Octahedron

To obtain a valid knot path, I first try to find a closed circuit on the polyhedron edges with as much symmetry as possible. An Eulerian circuit requires that all the vertices have even valence. Among the five Platonic solids, only the octahedron meets this condition (Figure 5a). To maximize symmetry, I regard the octahedron as a 3-sided anti-prism; this readily leads to a meandering path that exhibits D_{3d} glide symmetry (Figure 5b). To turn this Euler circuit into a knot, I must now avoid the intersections of the two passes that the knot strand makes through every vertex. A lopsided extension of all the lobes makes this possible, but it breaks all mirror symmetries. However, the same entanglement can be used at all six vertices, leading to an alternating knot that still exhibits D_3 symmetry (Figures 5c and 5d).



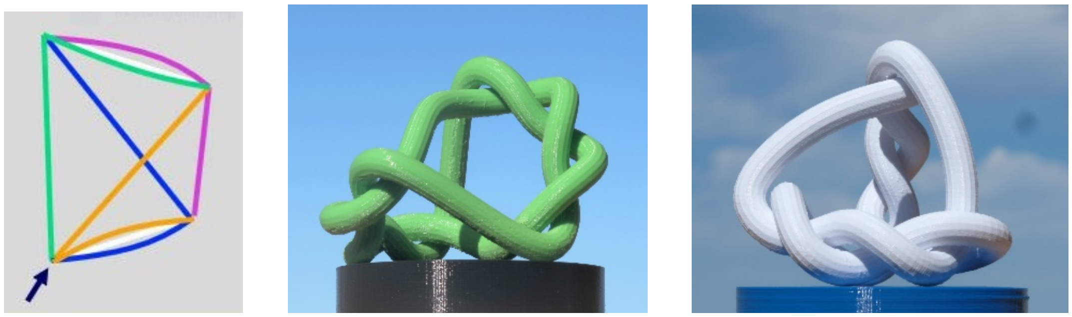
(a) (b) (c) (d)

Figure 5: *(a) Octahedron; (b) an Euler circuit on it; (c) wire-frame knot; (d) 3D-print.*

The Other Four Platonic Polyhedral Edge Knots

The other four Platonic solids have vertices with odd valences; therefore, an Eulerian circuit cannot be drawn on their edges. I remedy this problem by judiciously doubling some edges, so that all vertices assume an even valence. For a starting polyhedron with *v* vertices, I need to double a subset of *v*/2 edges in a configuration of maximal symmetry. On the resulting graph I can then draw an Euler circuit.

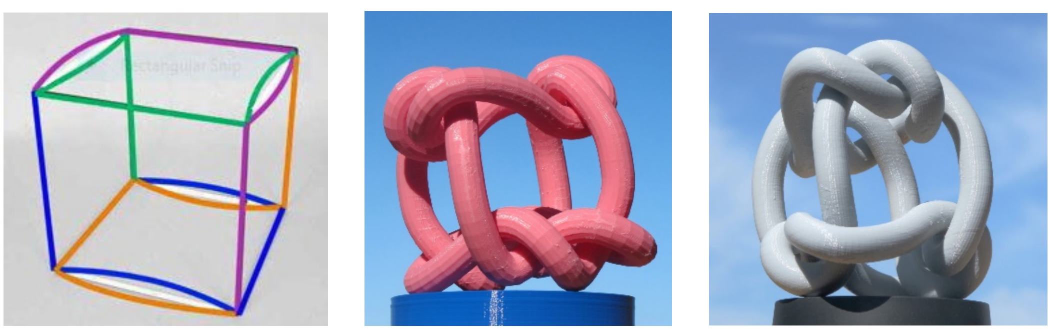
Figure 6a shows the chosen path for the *tetrahedron*: It starts at the arrow and follows green – orange – magenta – and blue, back to the starting point. Where the knot strand passes twice through each of the doubled edge pairs (magenta/green and yellow/blue, I let the two strands twist around each other in a helical manner. If the two double-strands twist in *same* way, then the result is a knot with D_2 symmetry (Figure 6b); this happens to be the non-alternating Knot 7_4. If both helices twist in *opposite* directions, one obtains the alternating Knot 8_3; it exhibits S_4 symmetry (Figure 6c).



(a) (b) (c)

Figure 6: *(a) Tetrahedron with two opposite edges doubled; (b) a resulting polyhedral edge knot: Knot 7_4 with D_2 symmetry; (c) a variant in which both helices twist in opposite directions: Knot 8_3.*

Next, I use the same approach for the *cube*. Here, four edges must be doubled so that all vertices have a valence of 4. When doubling a set of four parallel edges, resulting in a D_{4h}-symmetric configuration, I could not find a good Euler circuit of high symmetry. Instead, I double a set of edges as indicated in Figure 7a, exhibiting D_{4d} symmetry. I reversed the twists on the top and bottom pair of the doubled edges, to obtain a fully alternating knot with S_4-symmetry; it is the 12-crossing Knot $12a_{1288}$. When all helices twist in the same direction, the resulting knot is Knot $12n_{888}$, exhibiting D_2 symmetry (Figure 7c).



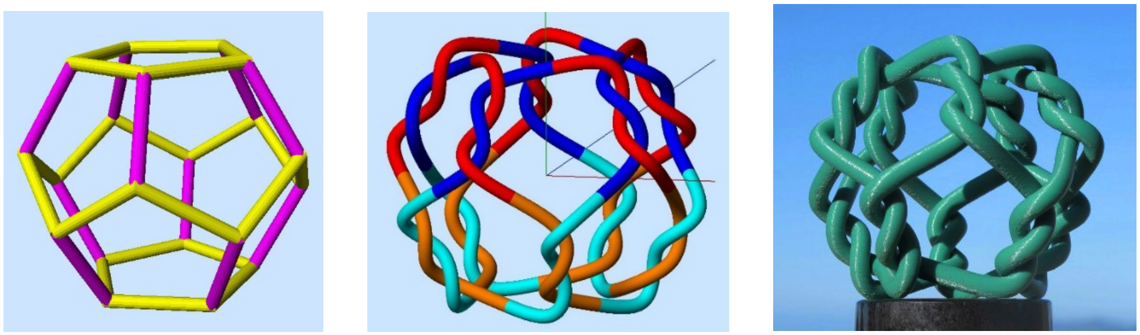
(a) (b) (c)

Figure 7: *(a) Cube with 4 doubled edges; (b) alternating Knot $12a_{1288}$, (c) non-alternating Knot $12n_{888}$.*

The *dodecahedron* has twenty valence-3 vertices; thus, ten edges (shown in purple) need to be traversed twice. As shown in Figure 8a, a nice D_{10d}-symmetrical solution can be found. Figure 8b shows a knot strand that maintains S_{10} symmetry. The corresponding 3D-print is shown in Figure 8c.

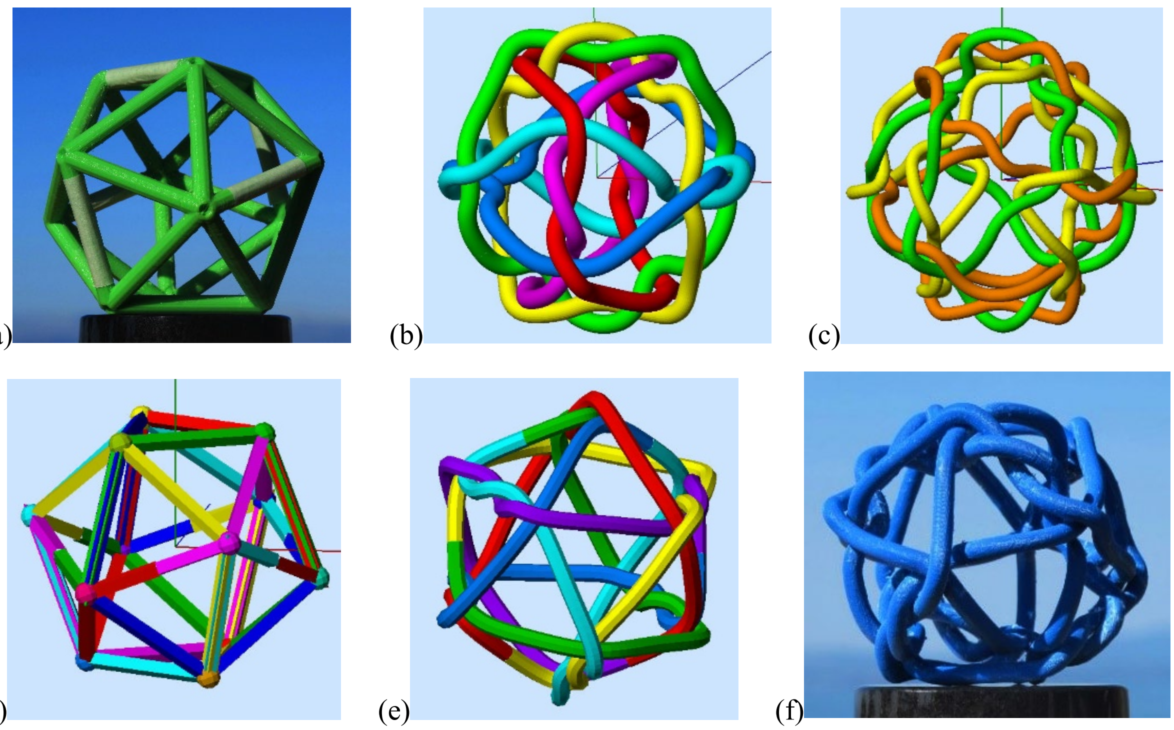
The *icosahedron* presents more difficulties. I first selected six mutually perpendicular edges (shown in grey) to be doubled (Figure 9a). However, when trying to find a highly symmetrical Euler circuit and corresponding knot strands, I ended up with six or with three congruent linked loops (Figures 9b and 9c).

Instead, when I chose a D_{3d}-subset of double edges (shown 2-colored in Figure 9d), I could find an Euler circuit that maintained S_6-symmetry. I was able to maintain this symmetry in a properly linked wire-frame knot. Figure 9e shows the complete knot strand composed of six congruent, differently colored, consecutive strand segments. Figure 9f shows a corresponding 3D-print. All these models were designed with Berkeley SLIDE [10].



(a) (b) (c)

Figure 8: *(a) Dodecahedron with 10 double-edges; (b) knot strand with S_{10} symmetry; (c) 3D-print.*



(a) (b) (c) (d) (e) (f)

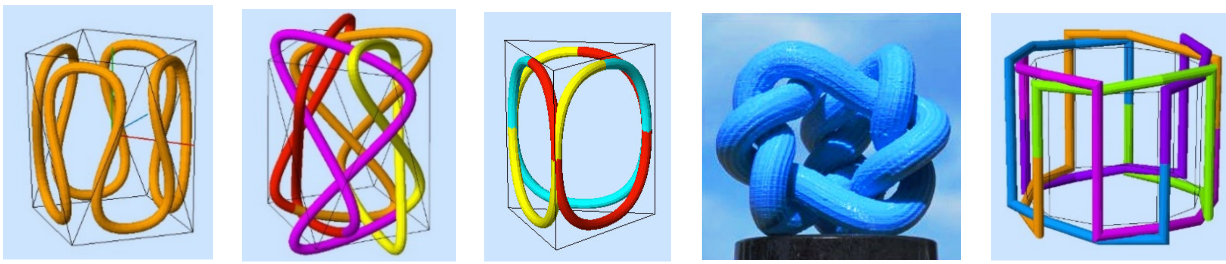
Figure 9: *(a) Icosahedron with 6 orthogonal edges marked for doubling. (b) Link of 6 congruent loops. (c) Link of 3 congruent loops. (d) D_{3d}-subset of double edges and corresponding S_6 Euler circuit; (e) corresponding S_6 wire-frame knot. (f) 3D-print.*

Prisms and Anti-Prisms

Of coures, we need not limit ourselves to start with one of the Platonic solids. We can start from any more or less symmetrical polyhedron – either an Archimedian solid or even a simple prism.

The octahedron, when seen as a 3-sided anti-prism, can serve as a guide for finding symmetrical Euclidian circuits for all other anti-prisms. All anti-prisms possess only vertices of valence 4; thus, no edge-doubling is required. All *s*-sided anti-prisms can be covered nicely with a path with D_{sd} symmetry that meanders up and down around all consecutive triangular side-faces (Figure 10a). This circuit can then be turned into a properly entangled knot with S_{2s} symmetry (Figure 10b) – as was done for the octahedron.

Straight prisms can be covered in a similar manner, if they have an *odd* number of sides. Figure 10c shows the general approach. Similar to the path shown in Figure 10a, the knot path travels up and down on the prismatic side-edges and, in between, advances one step around the prism along the edges of the end-faces. After *two* full tours around the prism, the path has traversed all end-face edges once, and all longitudinal prism edges exactly twice (Figure 10c). Along the latter edges, the two passing strands are helically wound around each other. Figure 10d shows the resulting sculpture for a 3-sided prism.



(a) (b) (c) (d) (e)

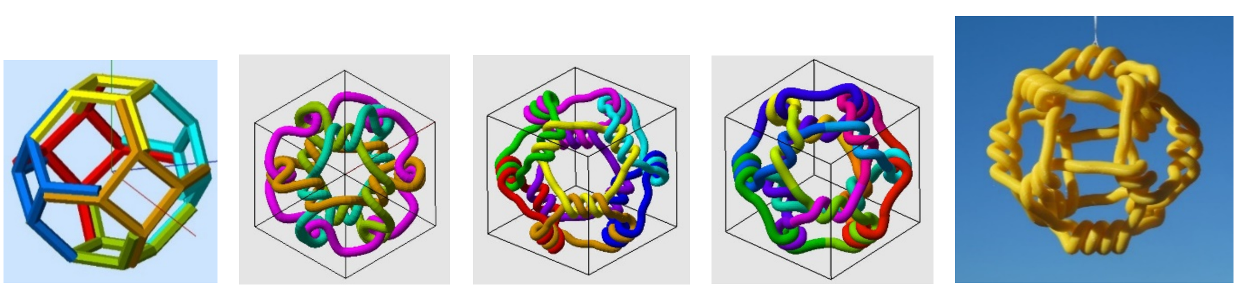
Figure 10: *(a) General anti-prism circuit. (b) 4-sided antiprism knot. (c) Circuit on a 3-sided prism; (d) corresponding prism sculpture. (e) Euler circuit on a 10-sided prism.*

Prisms with an *even* number of sides are harder to deal with. A cube can be seen as a 4-sided prism; thus the Eulerian circuit on the cube (Figure 7a) can serve as a guide how one might handle other even-sided prisms. On the cube, the path makes a vertical transition after taking *three* steps along the edges of an end-face. After four such moves, i.e., three full tours around the prism, the path has traversed all vertical edges once, and half of the end-face edges twice. Figure 10e shows how this approach can be generalized, using the example of a 10-sided prism. But this approach only works for two thirds of all cases. When the number of sides is divisible by six, the path closes on itself after just one tour around the prism, and this procedure would result in a link with three components. I have not yet found a general technique for handling this remaining third of the even-sided prismatic cases.

Archimedean Solids

Here is an example based on semi-regular polyhedra. The shapes in Figure 11 can either be seen as truncated octahedra with 12 edges doubled (Figure 11a) and with the two parts of the knot strand passing through those edges intertwined in a helical manner. Alternatively they can be seen as knots on the edges of a cuboctahedron with complex helical linkings at all 12 vertices, stretching them into edge-like structures.

If all these linkings are helices with 1.5 turns, the result is a link with four components, which roughly follow equatorial circles around the cuboctahedron (Figure 11b). Giving all helices a full number of twists, yields eight roughly hexagonal linked loops (Figure 11c). Both these link sculptures have the symmetry of the oriented octahedron. To turn this into a single knot, three of the helices need to make full twists, while the other nine helices make an odd number of half-twists (Figure 11d). This results in a single alternating knot, with D_3 symmetry, shown as a 3D print in Figure 11e.

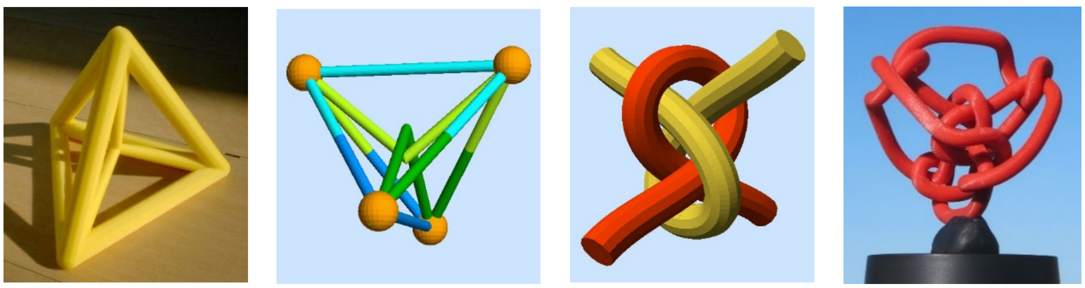


(a) (b) (c) (d) (e)

Figure 11: *(a) Truncated octahedron with doubled edges. (b) 4-component link. (c) 8-component link. (d) A single wire-frame knot with D_3 symmetry; (e) corresponding 3D-print.*

4. Graphs Derived from Regular 4D Polytopes

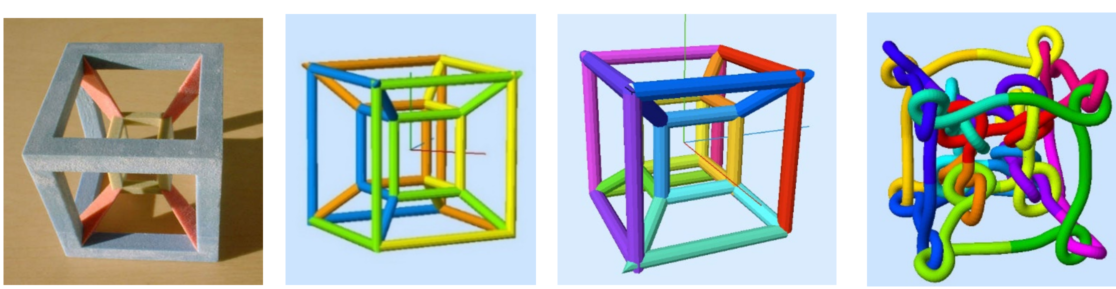
We don't have to restrict ourselves to the edges of convex polyhedra. Any nicely symmetrical graph with all even-valence nodes can serve as a good starting point for a knot sculpture. A first candidate that came to my mind is the cell-first projection of the 4D simplex into Euclidian 3-space. The edge graph of the 4D simplex can be depicted as a tetrahedral wireframe with an extra node in the center, which connects to the four tetrahedral vertices (Figure 12a). All nodes are valence 4. Now the challenges are to find an Eulerian circuit with maximal symmetry and then to find a way to loop the knot strand through the five nodes to maintain as much symmetry as possible. For the first task, we can take the tetrahedron with two doubled edges (Figure 6a) and pull one edge from the top and one from the bottom to cross and link in the center (Figure 12b). To make this an interesting knotted sculpture, I let these two edges form a *Granny-knot* in the center (Figure 12c). For the crossings of the strand in the *outer* four vertices, I simply let one strand make a loop around the other one. While the overall shape follows the symmetrical frame of the projected 4D simplex, I have not yet found a way to maintain any strict symmetry in the final knot (Figure 12d).



(a) (b) (c) (d)

Figure 12: *(a) 4D simplex projected into R3; (b) Euler circuit on this simplex. (c) The central Granny knot. (d) A resulting wire-frame knot sculpture.*

In the same spirit, I am trying to find a good solution for the Hypercube (Figure 13a). Its edge graph has 16 nodes of valence four; thus, no edge doubling is necessary. However, it is less obvious how to construct a good traversal through all the 32 edges that is as symmetrical as possible. If I try to maintain C_4 symmetry by repeating one path component four times around the z-axis, I obtain two separate circuits (Figure 13b). To obtain a single circuit, the symmetry needs to be broken; one component might make an extra quarter turn around the z-axis, while another one makes one quarter turn less (Figure 13c). Of course, the actual knot strand based on such an Euler circuit does not show any strict symmetry either (Figure 13d).



(a) (b) (c) (d)

Figure 13: *(a) 4D hypercube projected into R3. (b) Not quite an Euler Circuit; 2 components. (c) A proper Euler circuit with no symmetry; (d) an equivalent hypercube wire-frame knot.*

5. Discussion and Conclusions

Non-trivial prime knots can neither exhibit mirror symmetry nor the higher-order symmetries of the Platonic solids. However, tubular sculptures representing a single knot with a structure that mimics a Platonic solid can still be constructed by letting the knot strand follow the edges of such a regular polyhedron. The first step is to turn the given polyhedral edge graph into a graph with all vertices of even valence. Jablan et al. [6] describe several ways how this can be done. I have used selective edge-doubling, because it nicely preserves the desired edge geometry. The next, more difficult step is to find an Eulerian circuit on this graph. This circuit can then be converted into a mathematical knot by placing the appropriate Conway tangles [1] onto each valence-4 vertex. When strands running through a doubled edge are wound around each other, the resulting type of knot will change. This is of no concern to me, since my goal is not to generate any specific knot, but to design an overall pleasing geometrical object. This then leads to the most work-intensive phase: the construction of the sweep paths for the various helices and for the interlinked loops at all the vertices. Using judiciously chosen, parameterized control points for the various B-spline curve segments allows me to fine-tune the geometry while looking at the complete sculpture [10].

If maximizing symmetry is the main goal, then it is better to construct multi-component links, rather than single-thread knots. Mathematical links are *not* restricted in the symmetries they can exhibit. The simple Hopf link (Figure 14a), exhibits D_{2d} symmetry with two mirror planes. The same symmetry is also displayed by the four circular border curves in Perry's sculptures "Tetra" and "D_2d" (Figure 14b) [7].

(a) (b) (c) (d)

Figure 14: *(a) "Bonds of Friendship" by Robinson [8]. (b) "D_2d" by Perry [7]. (c) "Tetra Tangle" by Séquin [9][9]. (d) 20-component link with cube symmetry.*

With the use of mathematical links, rather than knots, it is easy to build sculptures with arbitrary symmetries, including the spherical symmetries of the Platonic solids. As an example, "Tetra-Tangle" is composed of four interlinked triangular frames (Figure 14c) [9]; it has the symmetry of the oriented octahedron. One can also readily construct a structure that has the full 48-fold symmetry of a cube. It may be composed of twelve tori representing the edges of a cube, and it may use another set of eight tori (shown in red) that hold together the edge-tori (Figure 14d).

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