Beyond the Great 96

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Abstract

Islamic geometric patterns often feature symmetric motifs like stars or rosettes. Historical designs include motifs with up to 96 points, but traditional construction techniques make it difficult to go higher. We present a simple polygons-in-contact method for constructing designs around a large central motif with any number of points. The method surrounds a large central polygon with conformally mapped layers of smaller tiles. The fidelity of the final design improves as the central motif grows.

Introduction

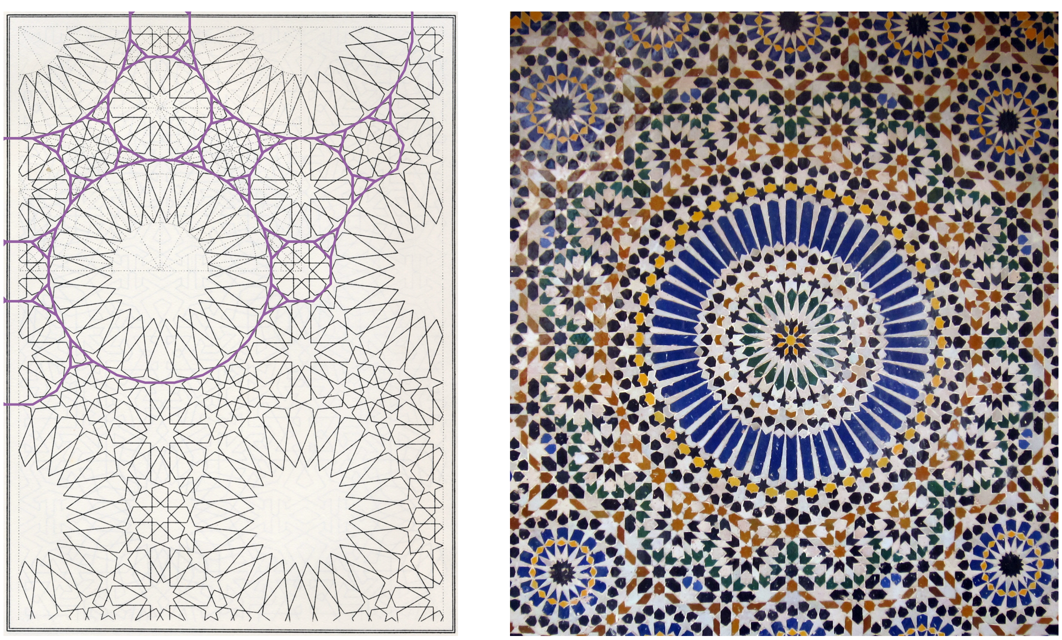

Figure 1: *Two historical examples of Islamic geometric patterns with large motifs. On the left, Plate 140 from Bourgoin [2], featuring rosettes with 8, 12, and 24 points. We superimpose part of the base tiling used in the polygons-in-contact method. On the right, part of a zellij mosaic from Fez, Morocco, with a central 48-pointed rosette.*

One of the most distinctive features of Islamic geometric patterns is the presence of star motifs with different numbers of points. A given pattern may be built from a regular arrangement of n-pointed stars for a single value of n, or from a combination of stars with different numbers of points. From an historical perspective, we may be interested in cataloguing the sizes—or combinations of sizes—that arise in the traditional practice of Islamic art. From a mathematical perspective, we wish to know which unused sizes or combinations might yet produce appealing new designs.

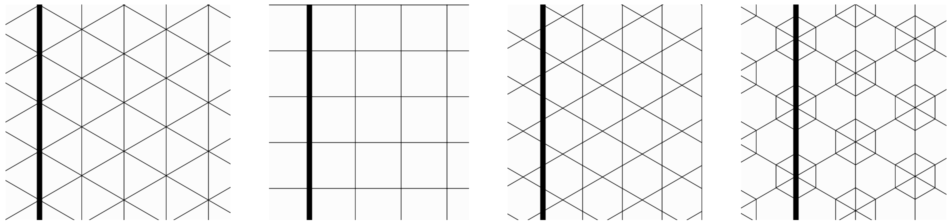

Figure 2: *Four examples of tilings by regular polygons that contain fault lines. For each, one vertical fault line is shown in bold.*

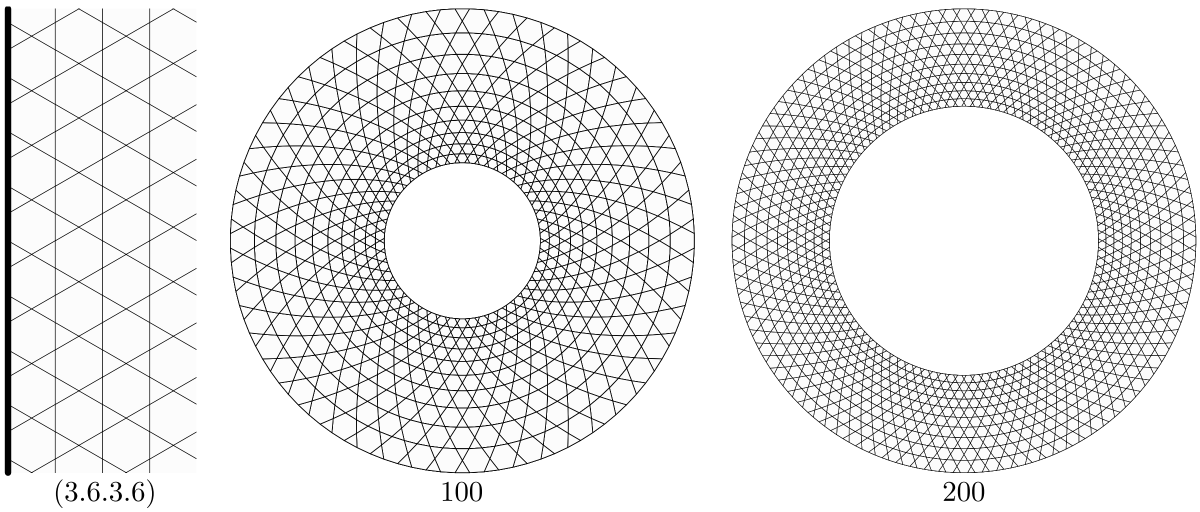

Figure 3: *On the left, part of the Archimedean tiling* (3.6.3.6)*, oriented so that a fault line is drawn vertically in bold. The centre and right drawings show ten layers of tiles wrapped conformally around a central* 100*-gon and* 200*-gon, respectively.*

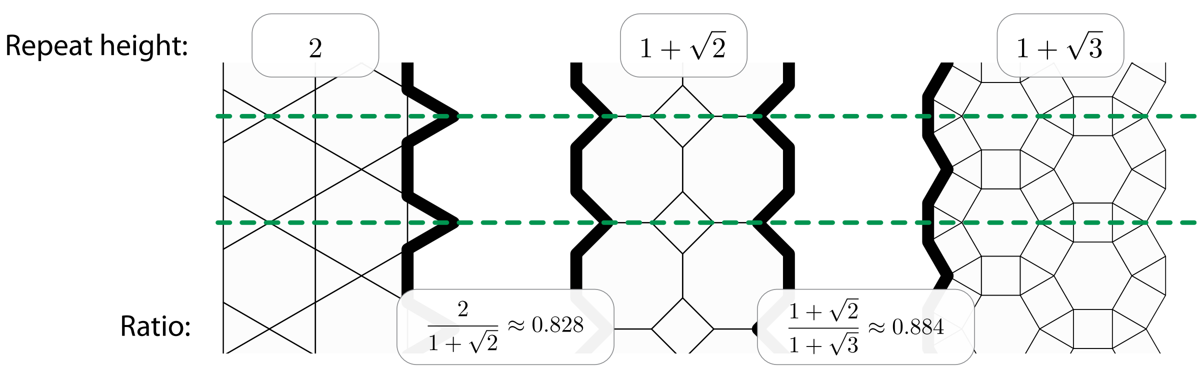

Figure 4: *Two examples of abutments of tilings that step down edge lengths. The strip between the horizontal lines shows one vertical repetition of the tilings. The repeat heights (top) give the height of the strip measured in terms of the edge length of each tiling. The ratios of these measurements (bottom) give the amounts by which edge lengths are stepped down across each abutment.*

For small values of n (up to about 24), the polygons-in-contact method [1, 7] provides a basis for productive exploration of patterns with n-pointed stars. In this method, motifs are inscribed in the tiles of a base tiling. Regular n-gon tiles are inscribed with n-pointed stars, or sometimes with more elaborate motifs such as rosettes [9]. The creation of new designs then rests on developing base tilings that feature many regular polygons. The regular and Archimedean tilings [6] immediately give us various combinations of stars with 3, 4, 6, 8, and 12 points. By introducing additional irregular tiles we can easily find and construct patterns with any value of n up to 16, and a few larger sizes [1]. Plate 140 of Bourgoin's collection [2] includes 24-pointed stars (Figure 1, left). Beyond that, the irregular tiles tend to become too unwieldy to support the construction of satisfying motifs consistent with Islamic design.

In Morocco, a separate tradition arose based on a system of *zellij* tile shapes, derived mostly from the geometry of a canonical eight-pointed star. Unlike polygons-in-contact, here a pattern is assembled directly from tiles. Zellij shapes are frequently arranged to form the backdrop for a monumental, many-pointed central star (Figure 1, right). Castera documents and reproduces designs with central n-pointed stars for most multiples of eight up to "The Great 96" [3, 4]. In unpublished notes and sketches, Tony Lee has produced his own designs with up to 120 points. The challenge in these designs is to reconcile the n-fold rotational symmetry of the central star with the natural eightfold symmetry of the ambient field of zellij shapes. The solution requires a "transition zone" of special-purpose tiles (including seven-pointed stars and other approximations of standard shapes) that absorb and distribute geometric error while bridging the gap. But this transition zone is increasingly difficult to construct as n grows.

Rather than striving upward incrementally in search of ever larger stars, our goal in this paper is to "come down from infinity". We recognize that as n grows very large, nearby points in an n-pointed star become approximately collinear. Beginning with this observation, we demonstrate a construction, based on polygons-in-contact, that could in principle be used to create designs with central stars of any size. Our construction surrounds a central regular n-gon with concentric layers of tiles, within which motifs may be inscribed. We show how to extend these layers of tiles to any desired depth while keeping their sizes (more precisely, their edge lengths) approximately constant.

Please visit isohedral.ca/great-96 for more information about this work, including an interactive tool for constructing large stars.

Surrounding the Central Star

Suppose that we wish to build a design around a central n-pointed star or rosette for some large value of n. Because we are using polygons-in-contact, we know at the outset that the central motif will be inscribed within a regular n-gon. For convenience, let this n-gon have unit-length edges. Our goal is to embed the n-gon within a larger field of tiles, whose inscribed motifs will combine with the centre's to yield a complete composition.

What tiles might we place adjacent to the edges and vertices of the central n-gon? Our philosophy of "coming down from infinity" suggests that we should treat n-gon vertices as subtending an angle of approximately 180 degrees. In that case, one solution would be to choose a tiling that contains "fault lines": infinite sequences of collinear, unit-length edges. We would then construct a strip of tiles rising from a fault line, and wrap the fault around the perimeter of the n-gon. Many tilings, including the regular tilings by squares and equilateral triangles, include fault lines; Figure 2 shows a few examples. Ideally, the tiling we choose should include larger regular polygons, which permit more interesting motifs.

We opt for the tiling shown on the left in Figure 3. It is known as (3.6.3.6), based on the arrangement of triangles and hexagons around each vertex. To create a continuous annulus of this tiling around a regular n-gon, first translate and rotate the tiling so that a fault line lies on the y axis of the plane. Scale the tiling so that its edges have length $2\pi/n$. Finally, draw as many columns of tiles as are desired, transforming each

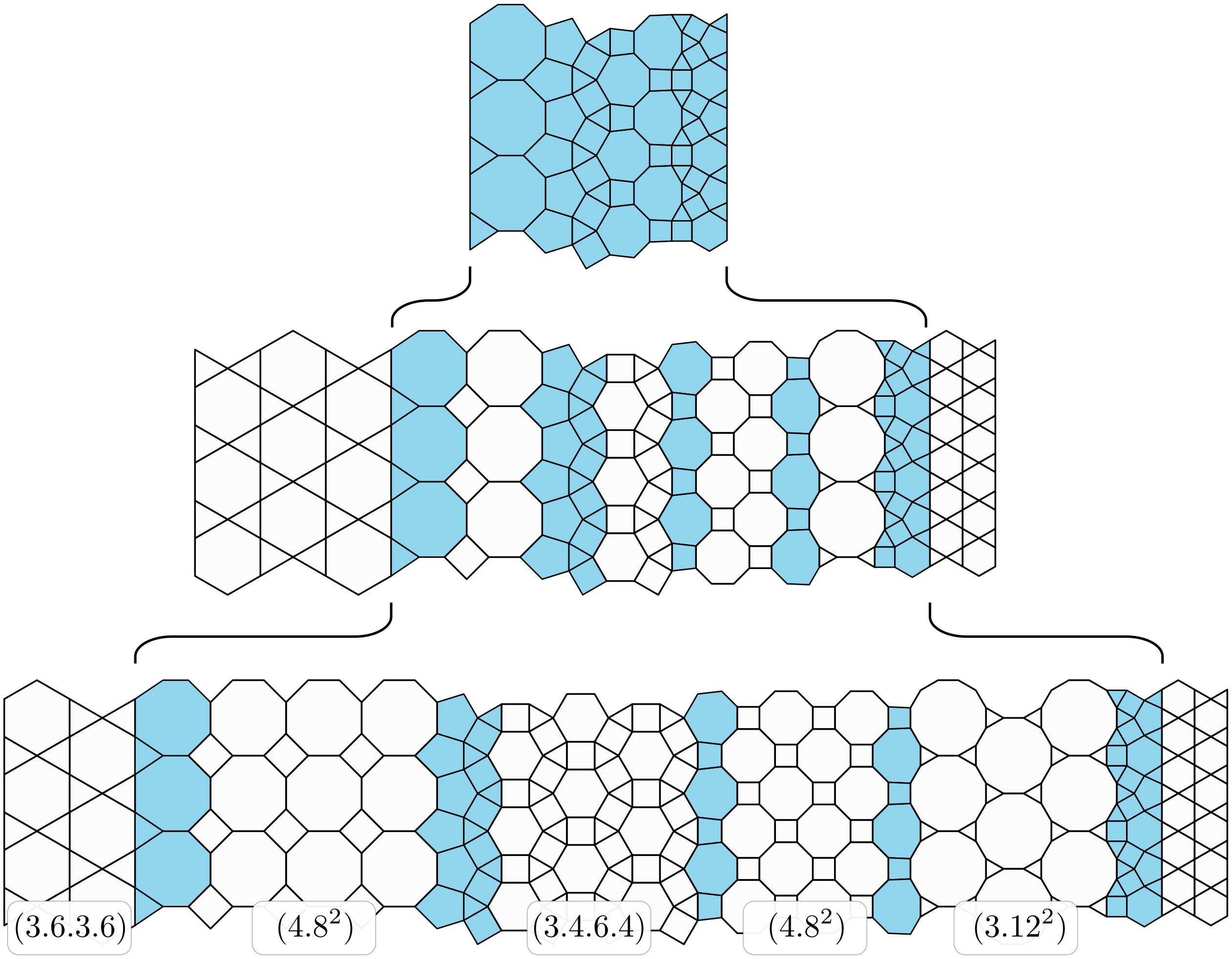

Figure 5: *A set of tiles that can effect a transition between two copies of the* (3.6.3.6) *tiling, with edge length* 1 *on the left and edge length* 1/2 *on the right. The centre row shows a typical transition, with irregular abutment tiles (shaded) separating regions of regular tiles. Regular tiles can be inserted or removed as needed to create transitions that are narrower (top) or wider (bottom).*

tile vertex (x, y) into $(re^x \cos y, re^x \sin y)$, where $r = 1/(2\sin\frac{\pi}{n})$ is the radius of a regular n-gon with unit edges. This final transformation is equivalent to exponentiation in the complex plane, a conformal mapping that Dixon referred to as the antiMercator map [5]. Note that n must be even in order to alternate triangles and hexagons along the n-gon's boundary. Figure 3 shows annuli of the transformed (3.6.3.6) tiling for two different values of n.

Combining Multiple Tilings

The construction of the previous section allows us to surround a central star with as many additional layers of motifs as we want, but it suffers from two shortcomings, both visible in Figure 3. First, an endless field of (3.6.3.6) does not offer much aesthetic variety. Second, the antiMercator map causes the edges of peripheral tiles grow exponentially without bound as we move away from the centre of the design.

To address both of these problems, we juxtapose columns of tiles from different tilings in a way that gradually steps down the lengths of tile edges, until we arrive at a copy of (3.6.3.6) with edges of length 1/2.

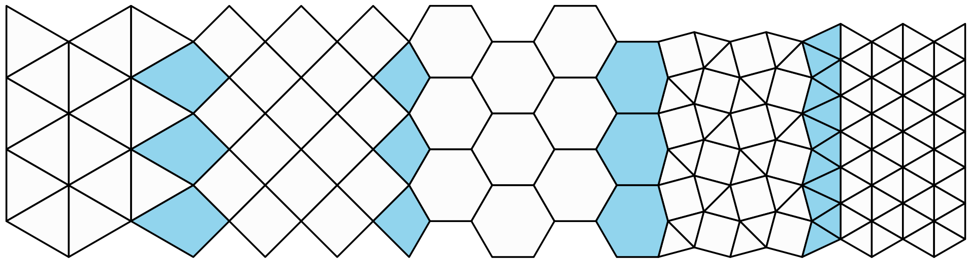

Figure 6: *A transition sequence based on the regular tiling by triangles, which can be used to double a central polygon with an odd number of sides.*

These juxtapositions use what Kaplan referred to as "abutment" [8]: we transition abruptly between tilings, interposing a few irregular polygons as a compromise where tile shapes would otherwise be incompatible. If we combine these transitions into a strip of tiles of the correct width, the halving of the edge lengths will be cancelled out by the scaling induced by the antiMercator map, and edge lengths of peripheral tiles will be kept approximately constant as we move away from the centre of the design.

We have developed a number of these abutments through trial and error. Two examples are shown in Figure 4. In an abutment, the two tilings that meet must share the same vertical translational symmetry. We can measure the length of that translation vector in terms of the edge lengths of each of the tilings. The ratio of those two measurements gives the amount by which the abutment steps down edge lengths. Ratios closer to 1 give smoother transitions. By chaining together a sequence of abutments, we can accumulate these ratios to give a product of $1/2$.

Figure 5 (centre) shows a complete transition sequence we designed, leading to a half-scale (3.6.3.6) tiling. The transition uses the abutments from Figure 4 as well as a few others, with columns of irregular tiles shown shaded in blue between them. Any such transition sequence can telescope: columns of regular tiles can be removed to create a smaller transition (top), or extra columns can be inserted to pad it out (bottom). In this way we can extend the width of the sequence to be as large as we want.

We can compute the ideal width of a transition sequence based on n. We would like the scale factor implied by the antiMercator map to be 2 at the right edge of the transition, meaning that after scaling the initial tiling by $2\pi/n$ and before exponentiating, the transition should stretch from $x = 0$ to $x = \ln 2$. In the unscaled tiling, with the tiles on the left having unit length, we would therefore need the whole sequence to have a width near $\frac{n \ln 2}{2\pi}$. For a given n, we can add or remove tiles as needed to approximate this width. Working backwards, we also see that the minimal transition at the top of Figure 5, with width around 6.81, would be ideal if n is around 62. The middle and bottom rows in the figure would correspond to values of n around 184 and 310, respectively. Note also that the outer boundary of a transition sequence is a regular $2n$-gon, allowing us to chain together as many instances of the sequence as we like to grow the design away from the central motif.

Because the fault line in a (3.6.3.6) tiling alternates triangle and hexagon edges, this transition requires n to be even if the concentric tiles are to wrap around the n-gon cleanly. For this reason we prefer large stars with even n. Given an odd value of n, we can first use a transition sequence like that of Figure 6 as a gadget to double n, and then revert to our (3.6.3.6) transition sequence for subsequent doublings.

Completing the Design

We can now use standard polygons-in-contact techniques to produce motifs for the polygonal tiles defined in the transition sequence above [7]. As always, for tiles that are regular polygons, the construction process is

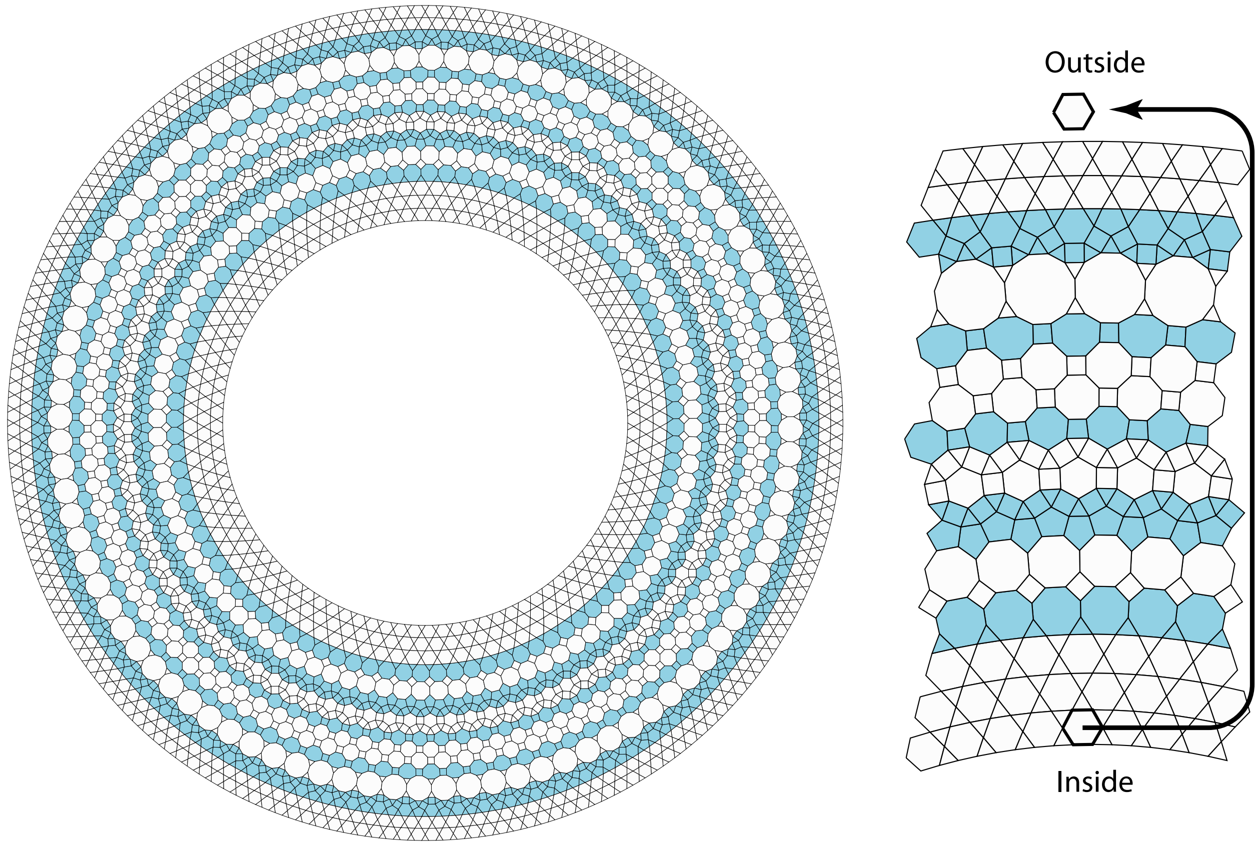

Figure 7: *An annulus of tiles layered concentrically around a central* 184*-gon. The transition sequence is based on the one shown in the centre row of Figure 5. On the right, we show an enlarged core sample of the annular mapping. A single tile from the inner boundary is copied to the outer boundary, to show that edge lengths are approximately preserved.*

straightforward and we can produce stars or rosettes directly. Irregular tiles require a heuristic process for building motifs inward from edge centres. In our implementation we compute irregular motifs automatically, though it may be possible to improve some of them through manual adjustment.

Once we have defined motifs for all tile shapes, we can arrange them in the same configuration as the transition sequences shown above, and transform them into place around a central regular n-gon using the antiMercator map. Note that drawing motifs before transforming them yields more faithful results than constructing motifs in tiles that have already been mapped concentrically. We complete the design by computing a large star or rosette for the central n-gon.

Figure 7 illustrates the construction of a tiling around a central 184-gon, based on the transition sequence shown in the centre row of Figure 5. Figure 8 shows the construction of motifs for the tiles of this transition sequence, and the creation of a final design based around a central 184-pointed rosette. Even with only 184-fold symmetry, the individual elements of the final design are difficult to resolve at the scale shown in this paper, though the use of multiple distinct base tilings lends the result an appealing textural variation. Designs with large stars are clearly best deployed at monumental scale. If the hexagons ringing the central rosette were manufactured with a diameter of one centimetre, then the entire design would be approximately two metres across. Designs based on central motifs with larger values of n are equally easy to construct, but effectively impossible to illustrate reliably in this format.

The supplementary information for this paper includes a PDF of an example based on a central 300-pointed star, which can be examined under different levels of magnification.

Conclusion

Beginning with the observation that the boundary of a regular n-gon becomes flatter as n grows large, we have presented a technique for constructing base tilings that give rise to Islamic geometric patterns with layers of motifs arranged concentrically around a central star or rosette. The process depends on the development of transition sequences in which we gradually step down the lengths of tile edges through a series of abutments, cancelling out the scaling induced by the concentric mapping. The width of the transition sequence gives a lower bound on values of n for which the mapping roughly preserves edge lengths in tiles. Any transition sequence can be telescoped outward, easily adapting to larger values of n.

These concentric designs have a high degree of rotational symmetry. If we wish to incorporate such a design into a larger periodic arrangement of motifs, we are faced with the same challenges seen in traditional zellij design: we will have to construct a specialized transition zone to connect our concentric tiles to an ambient periodic tiling. We hope to develop further techniques for such transitions in future work.

Acknowledgements

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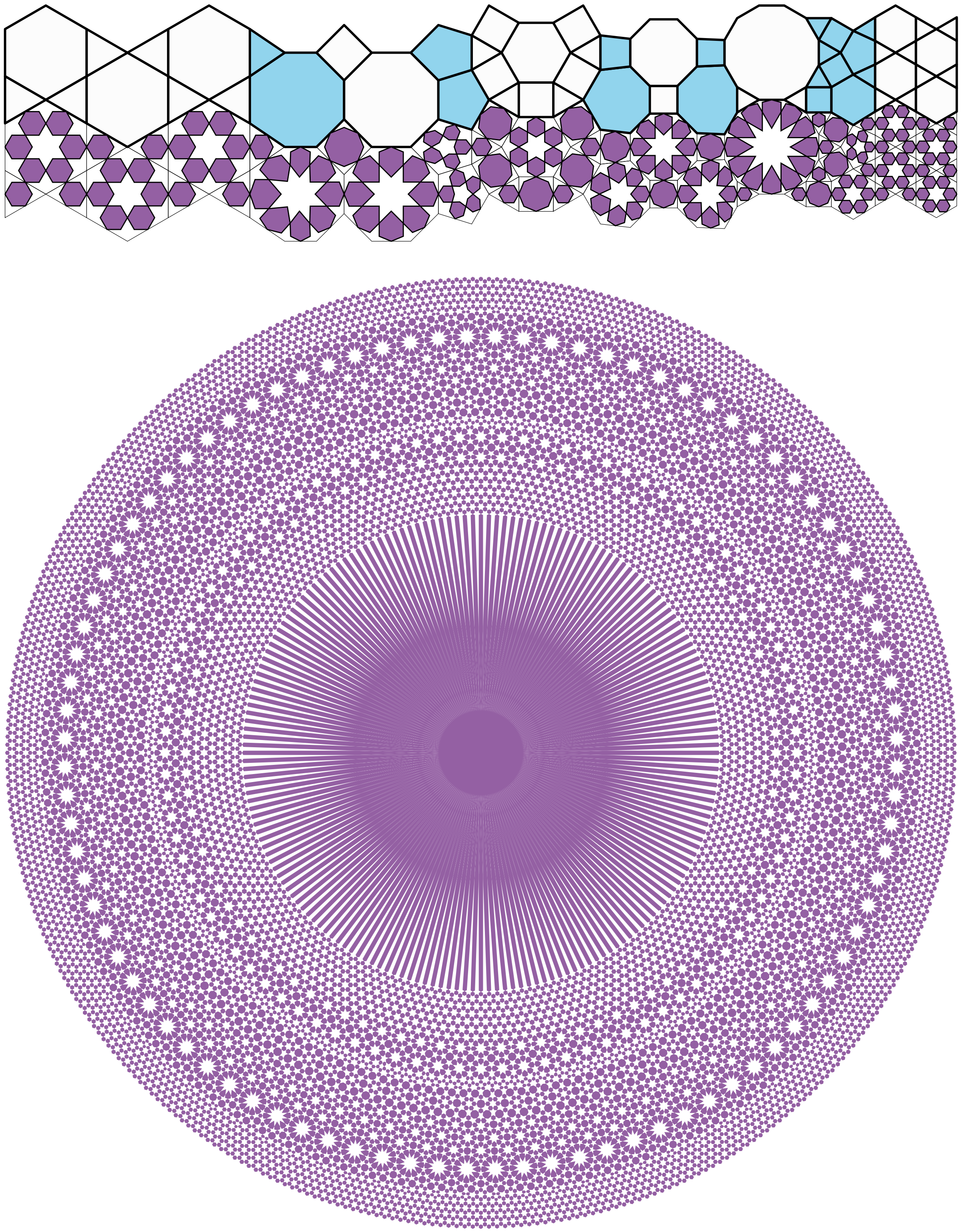

Figure 8: *Top, the transition sequence from Figure 5, where the lower row of tiles has inscribed rosettes and other motifs are constructed via polygons-in-contact. Bottom, these motifs are arranged concentrically around a central* 184*-gon to create a complete Islamic geometric pattern.*