Polyhedral Approximations of the Sphere in LEGO®

Jin Yamauchi[1] and Chamberlain Fong[2]

[1]Tokyo, Japan; jin_yamauchi_1112@lilac.plala.or.jp
[2]San Francisco, California, USA; chamberlain@alum.berkeley.edu

Abstract

We present our attempts at modeling the sphere using LEGO® parts. We approach this task by approximating various well-known polyhedral shapes and assessing their resemblance to the sphere.

Introduction

There has been considerable interest in modeling the sphere in LEGO®. In fact, LEGO® has released several retail toy products where the sphere plays a central role. Examples of these include toy planets, architectural domes, and even robots [3]. There is also a fan-built model known as the Lowell sphere [4] that is popular among enthusiasts in the LEGO® community. In this paper, we will present our alternative approach to modeling the sphere. Our approach is based on using well-known polyhedra that closely resemble the sphere. We shall a present a methodology for building such polyhedral shapes. To showcase our results, three Goldberg polyhedra that have reasonable semblance to the sphere are shown in Figure 1.

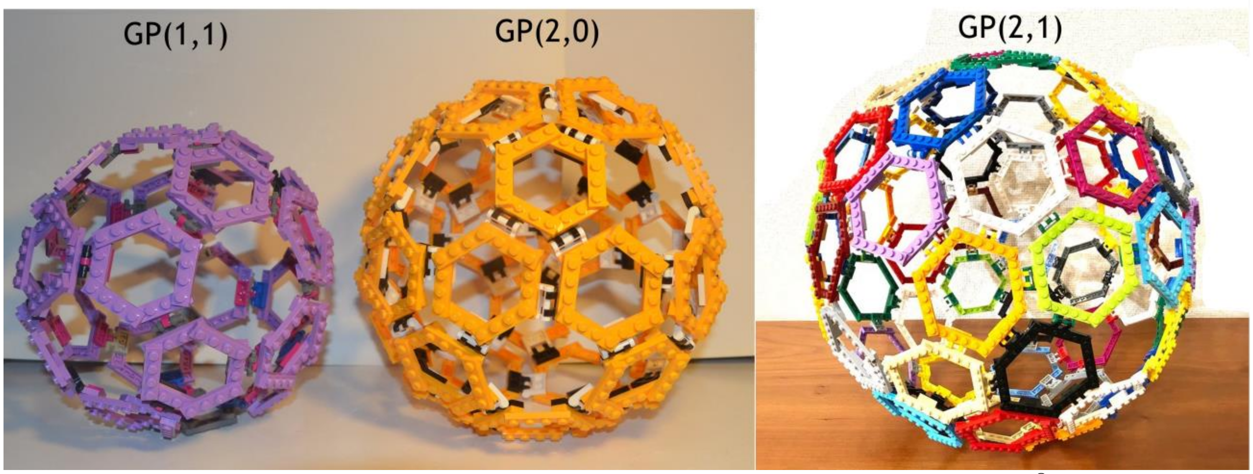


Figure 1: *Using Goldberg polyhedra to approximate spheres in LEGO®.*

We judge the effectiveness of a polyhedral approximation based on two criteria. The first criterion is the simplicity of the model. This can be measured by simply counting the number of parts used in the model. The second criterion is a measure of how closely the polyhedral shape resembles the sphere. For this, we use the concept of sphericity introduced by Hakon Wadell [5]. The sphericity ψ of a shape has the formula:

$$\psi = \frac{(36\pi V^2)^{\frac{1}{3}}}{A} \qquad \text{where V is the volume and A is the surface area of the 3D shape}$$

The sphere has a sphericity value of 1. A polyhedron that nearly resembles the sphere will have a sphericity value that is close to 1. For example, the GP(1,1) polyhedron, better known as the truncated icosahedron, has sphericity $\psi \approx 0.9666$. Intuitively, one can consider the sphericity as a numerical measure of 3D roundness.

LEGO® for Geometers

LEGO® was not designed for modeling polyhedra. In contrast, other interlocking plastic toys such as Zometool, Polydron, and Magna-Tiles were specifically designed for building polyhedral shapes. Nevertheless, LEGO® is versatile enough that it can be used to model some polyhedra. Furthermore, LEGO® pieces come in a wide variety of color, which make them quite attractive for sculpting geometry. In this section, we shall present our methodology for building polyhedra in LEGO®.

Although LEGO® has a vast number of parts in its inventory, we shall restrict our polyhedra to only use a few basic types. Specifically, we shall restrict the builds to using LEGO® plates, clips & handles. The reasoning behind this is twofold. We want to keep the builds simple. We also want to use parts that are common and readily available.

LEGO® plates are thin rectangular elements that can interlock with other parts. Some examples of common plates are shown in Figure 2. In our polyhedral modeling, plates serve as polygonal edges or faces. For example, a hexagon assembled from six 1x4 plates is shown in Figure 3. These plates join together at their ends using standard LEGO® connector studs. Note that the resulting hexagon is equilateral but not necessarily equiangular.

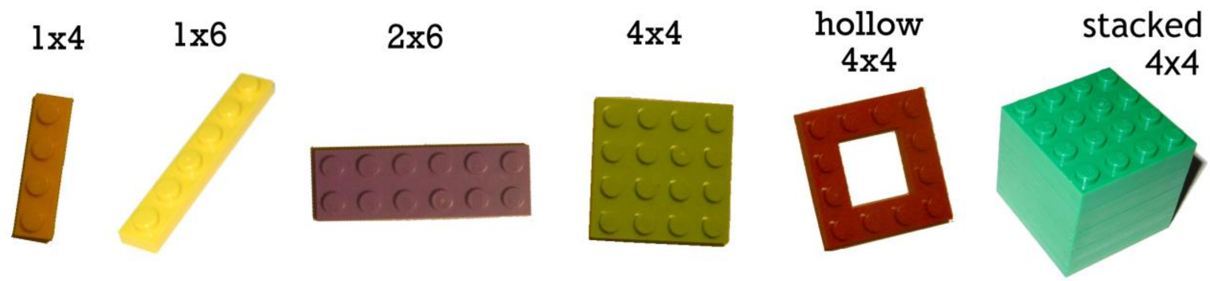


Figure 2: *LEGO® plates come in different shapes and sizes.*

Multiple polygonal faces in our models are joined together via LEGO® parts known as the clip and the handle. The clip is a plate with a U-shaped protrusion used for gripping. The handle is a plate with a cylindrical protrusion onto which the clip can fasten. Both parts are shown in Figure 3. The clip and handle combine to form a hinging joint, whereupon they fasten together at a wide range of angles. This angular versatility allows for the joining of faces at arbitrary dihedral angles in the polyhedron. For example, two hexagons joining at a dihedral angle of 150º are shown in Figure 3.

By just using many LEGO® plates, clips, and handles, we are able to build the three Goldberg polyhedra shown in Figure 1. Note that pentagonal faces of the polyhedra are only represented implicitly via the outlines of the adjoining hexagons. This is because our methodology is limited to polygons with an even number of sides. The limitation arises from the fact that we connect plates over and under each other in order to approximate a polygon. A polygon with an odd number of sides is not quite amenable to this kind of layered arrangement of LEGO® plates in a loop.

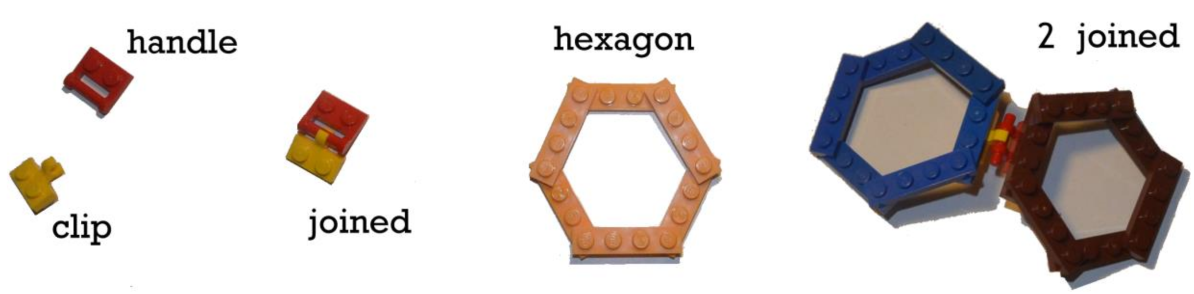


Figure 3: *Using clips & handles to join polygons at arbitrary dihedral angles.*

Polyhedral Approximations

Goldberg Polyhedra

Goldberg polyhedra [2] are an infinite family of polyhedra consisting of 12 pentagons and numerous hexagons. We already saw three examples of Goldberg polyhedra in Figure 1. It is important to note that our LEGO® models of Goldberg polyhedra are just approximations. For one thing, the hexagons built out of layered plates are not exactly planar. Also, there are significant spacing gaps between adjacent hexagons that are physically occupied by hinging joints. This means that edge lengths, distances, and symmetries within our models are off from the actual polyhedra. It is suffice to say that our models are topologically inspired from Goldberg polyhedra but are not quite mathematically precise. These inaccuracies carry over to corresponding sphericities.

It is possible to build larger polyhedra using our methodology. However, the resulting shape will likely be unstable and collapse under its own weight. In fact, the clutch power for the LEGO® connections in our GP(2,1) model was barely enough to overcome the pull of gravity and the whole structure was quite fragile. This problem can be remedied by using additional adhesives such as superglue or duct tape, but our aim is to determine what can be done with straight LEGO®, without resorting to such tactics.

In addition to simplicity and sphericity, we consider the stability of a model as highly desirable. There are two types of stability: static and dynamic. The former has to do the freestanding structural robustness of the model and the latter has to do a rolling ability of the model. Static stability is important for displaying the model. Dynamic stability is important for playing with the model.

Cube

It is possible to use our methodology to build a LEGO® cube with numerous plates and hinging joints; however this might be a bit of overkill. LEGO® was specifically designed for modeling shapes like the cube and it is much simpler to just stack multiple square plates to form a cube. This is shown in the rightmost part of Figure 2. In terms of structural stability, this rendition of the cube is rock-solid. Meanwhile, a cube built out of plates with 90º hinging joints can be stable enough to roll, but it is not as sturdy. Anyhow, regardless of its stability, the cube is a very poor approximation of the sphere. In fact, the cube and the sphere are so dissimilar that they can be considered as opposite sides of a spectrum [1].

Truncated Octahedron

Having exhausted small Goldberg polyhedra, we investigated other polyhedra with a small number of faces, which led us to the truncated octahedron. This Archimedean solid consists of 8 regular hexagons and 6 squares. Our LEGO® model of this shape is shown in Figure 4a. It uses far fewer parts when compared to the Goldberg polyhedra previously discussed. However, when it comes to roundness, the shape is a poor approximation of the sphere. Consequently, the truncated octahedron does not roll well, but it does roll better than the cube.

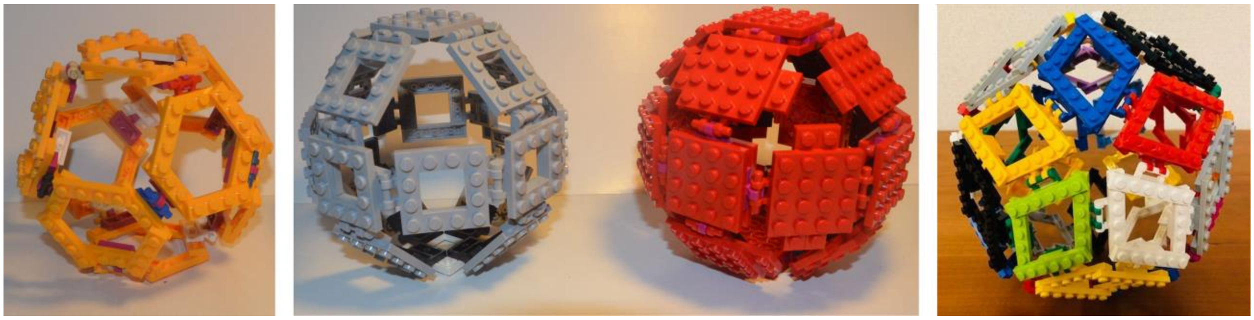
Rhombicuboctahedron

Our continued attempts to approximate the sphere brought us to another well-known Archimedean solid. The rhombicuboctahedron consists of 18 squares and 8 regular triangles. It can be modeled in LEGO® by using solid or hollow square plates as polyhedral faces. These are shown in Figure 4b.

We built two different models of the rhombicuboctahedron. The model on the left of Figure 4b uses hollow 4x4 square plates and has the appearance of a polyhedral skeletal frame. The model on the right of Figure 4b uses solid 4x4 square plates and has the fake appearance of a solid ball. There are additional 2x6 back plates attached as a stand-in for the triangles. These extra back plates make it sturdy enough to roll gently. Most of the other models in this paper will come apart when rolled.

Rhombic Triacontahedron

Our final approximation of the sphere in this paper comes in the form of a popular Catalan solid. The rhombic triacontahedron consists of 30 golden rhombi. It has a sphericity of 0.9609, which is just a tad off from the truncated icosahedron. Our LEGO® model of this shape is shown in Figure 4c.



(a) (b) (c)

Figure 4: *But will these shapes roll? (a) truncated octahedron, (b) rhombicuboctahedron, (c) rhombic triacontahedron.*

The parts count and sphericity values for various polyhedra used in this paper are tabulated in Table 1. The corresponding dihedral angles for clip & handle pairs are also tabulated. In terms of approximating the sphere, Goldberg polyhedra consistently provide high quality spherical models. However, they also require much more parts to build. Meanwhile, the truncated octahedron and the rhombicuboctahedron require far fewer parts to build, but the rhombicuboctahedron gives a better approximation of the sphere.

Table 1: *Sphericity and Parts Count for Various Polyhedra.*

polyhedron	other notes	sphericity	plate count	clip/handle pairs	dihedral angle
truncated icosahedron	GP(1,1); has 20 hexagons	0.9666	120	30	138.189º
chamfered dodecahedron	GP(2,0); has 30 hexagons	0.975	180	60	144º
truncated pentagonal hexecontahedron	GP(2,1); has 60 hexagons	0.982	360	150	153.178º
cube *(hinged version)*	6 faces and 12 edges	0.806	6	12	90º
truncated octahedron	Archimedean solid	0.9099	48	12	109.47º
rhombicuboctahedron *(latter version)*	Archimedean solid	0.954	36	24	135º
rhombic triacontahedron	Catalan solid	0.9609	120	60	144º

Summary and Conclusions

We explored LEGO® as a possible medium for modeling polyhedra and presented various polyhedral approximations of the sphere. Since LEGO® is a popular toy among children, it can serve as a gateway to fostering interest in mathematics.

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