Using Inflation to Lay a P3 Tiling in Two Dimensions and Three Dimensions

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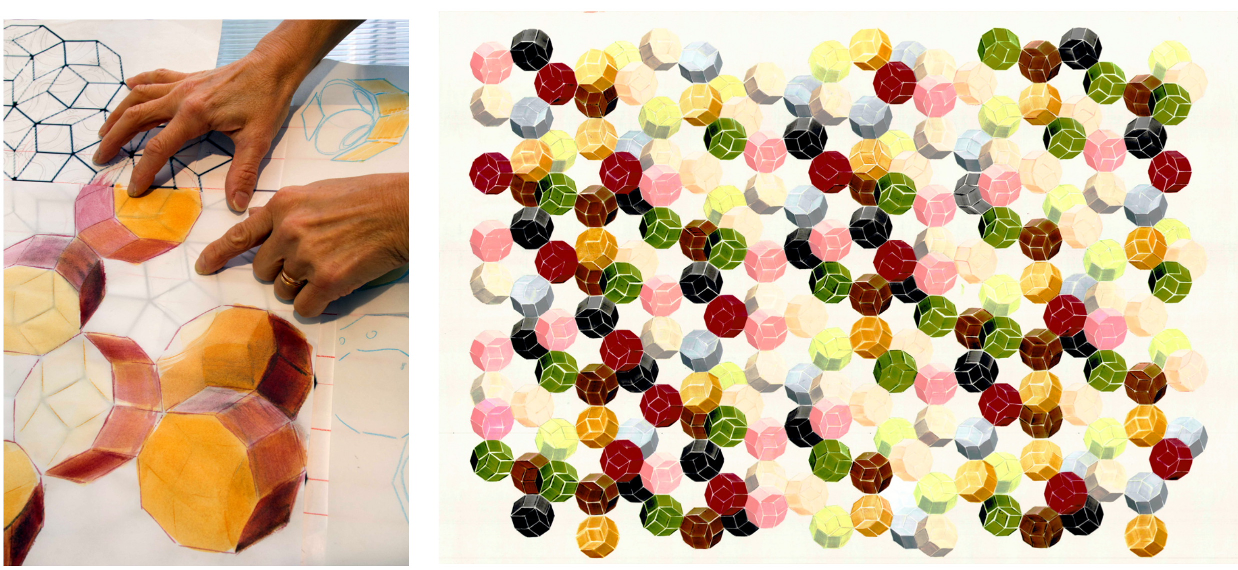
Abstract

This paper introduces a pair of crenellated fractal pavers made from an inflation of a Penrose tiling that were discovered exclusively through drawing and making sculpture. Pavers assemble according to local matching rules embedded into their perimeters, and may be used to construct two-dimensional and three-dimensional Penrose tilings by rhombs.

Introduction

A Penrose tiling by rhombs, or P3 tiling, is an aperiodic arrangement of two different rhombs that require special matching rules to assemble. Matching rules alone cannot guarantee an infinite tiling of the plane [4]. For this reason large areas of tiling are often generated by a process called *deflation* to avoid mistakes [1]. Deflation is an operation whereby old rhombs are subdivided into patches of new smaller rhombs. The procedural opposite to this is an inflation, the mapping of the tiling onto itself at a larger scale [2].



(a) (b)

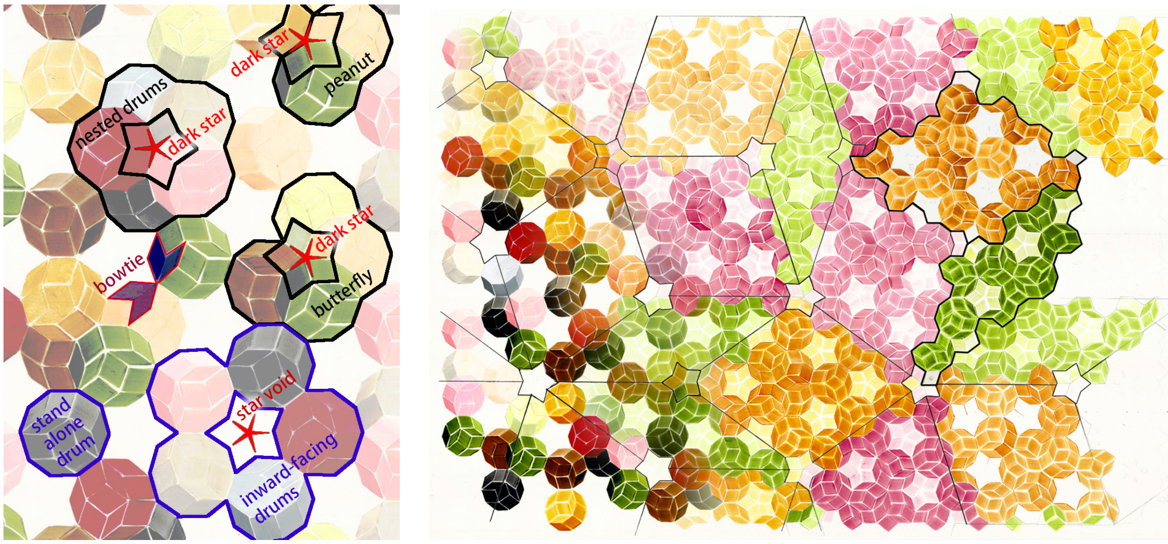
Figure 1: *(a) Configurations of ten rhombs painted to look like drums. (b) Drums in a 4,000-rhomb P3 tiling with each of the ten possible drum orientations painted a different color.*

One Pair of Two-Dimensional Pavers

I wanted to lay a large pavement of P3 tiling efficiently without having to lay rhombs individually like cobblestones. I imagined casting pavers from large patches of rhombs, and searched the tiling for a possible method. Like others before me I was intrigued by a configuration of ten rhombs set within a regular decagon, as in Figure 1a. Most circular motifs in a P3 tiling have five-fold symmetry. These do not; they look like drums.

Imaginary drums

I painted the decagons to look like real drums in perspective with light falling on them. Each drum has an apparent front, the skin of the drum, implied by a single axis of mirror symmetry. This front may be used to orient a drum within a tiling. There are ten possible drum orientations. Figure 1b shows every drum in a 4,000-rhomb P3 tiling; each is painted in one of ten different colors to call attention to it's orientation.



(a) (b)

Figure 2: (a) *Drum clusters, dark stars, star voids, and bowtie. b) The inflation and one pair of pavers.*

The drums in a P3 tiling may stand alone or partially overlap other drums. Approximately 55.5% of the drums in a tiling overlap to form clusters of 2, 4, or 5 drums. The three different types of overlapping cluster are outlined in black in Figure 2a. I call them *peanut*, *butterfly,* and *nested drum* clusters. Peanut and butterfly clusters have a single axis of mirror symmetry. Nested drum clusters have five-fold mirror and rotational symmetry. The fronts of drums in peanut, butterfly, and nested drum clusters face out from the center.

Non-overlapping drums may either *stand-alone*, or be positioned in rings of five *inward-facing* drums. These are outlined in blue in figure 2a. Inward-facing drum clusters have five-fold mirror and rotational symmetry. The fronts of a ring of inward-facing drums face toward the center of the cluster.

Dark Stars and Star Voids

Painting all the drums in a P3 tiling reveals a pattern of *star voids* that cannot be covered by any drum. Rings of inward-facing drums always surround a star void. Nested drums and butterfly clusters have patches of five fat rhombs that are the same shape as star voids. I call these *dark stars* to distinguish between the two types of star. Dark stars and star voids would look identical if no drums were colored.

In both types of star the edges of five fat rhombs meet at a central vertex. These vertices are marked as small red asterisks in Figure 2a. In relation to the drum clusters that cover a P3 tiling, dark stars are only found at the three locations marked in black in Figure 2a. All unpainted areas in Figure 2a are star voids.

A Natural Inflation

A natural inflation emerges when rings of inward-facing drums and clusters of nested drums are connected by straight lines. This inflation is indicated by thin black lines in Figure 2b. There is no ambiguity to the location of the vertices that give rise to this inflation, and no exceptions to the following rules: All nested drum clusters and rings of inward-facing drums present in the original P3 tiling will become vertices; There are no vertices in the inflation that are *not* surrounded by either inward-facing drums or nested drums; The vertices of the inflation align precisely over the central vertex of either star void or dark star; However, of the countless star voids and dark stars in a P3 tiling, only those located inside nested drum clusters or within rings of inward-facing drums will give rise to this inflation.

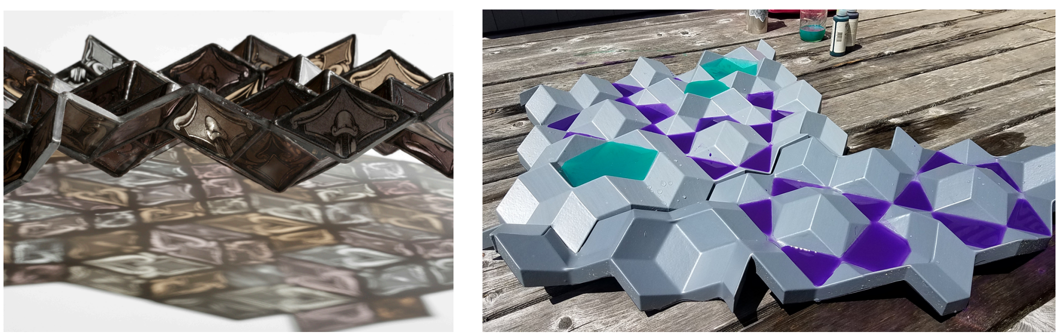
I used this inflation to design a pair of pavers that correlate to Penrose's thick and thin rhombs with local matching rules embedded into their perimeters.

Designing the Pavers

I designed the perimeters of the pavers using the following rules applied sequentially: Keep drums intact where possible; Never divide one-half of a bowtie. A bowtie may be seen outlined in red in Figure 2a; When an inflation line cuts through a stand-alone drum keep the front of the drum intact and bisect its edge; At vertices, use rotational symmetry to divide rhombs equally between adjacent pavers; When an inflation line cuts through a peanut keep one of the two drums intact. One thick and one thin paver may be seen outlined in heavy black lines in Figure 2b.

Two Pairs of 3D Pavers

A three-dimensional P3 tiling is an assembly of identical golden rhombs that meet edge to edge. I began making sculptures of P3 tilings in 2014 in a variety of materials using a diagram from a 1981 paper by N. G. DeBruijn as a guide [3]. These sculptures require only one rhomb, a golden rhombus tilted to project a shadow of either thick or thin rhomb onto a single plane. Thus a Penrose tiling may be thought of as a drawing or photograph. It is a two-dimensional image of a three-dimensional object. Figure 3a shows a three-dimensional P3 tiling made in stained glass with projected shadow.



(a) (b)

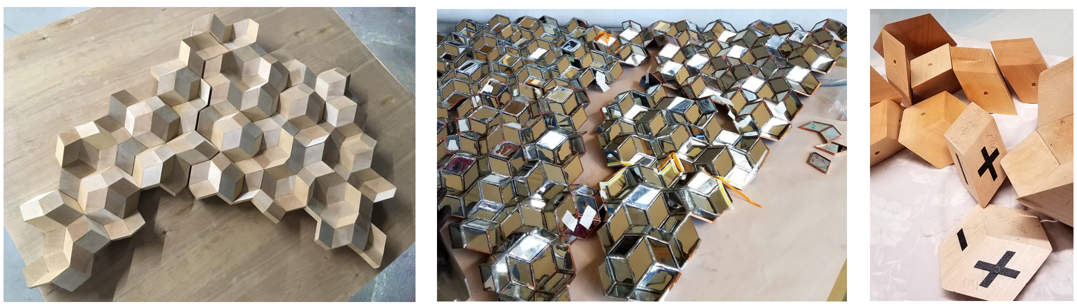
Figure 3: *Three-dimensional P3 tilings: a) stained glass with projection, b) vacuum-formed plastic molds with colored water to indicate vertex heights.*

All vertices in a three-dimensional P3 tiling are positioned at one of four discrete heights. Rhombs assemble as a crystalline layer confined between parallel planes. I imagine these as the floor and ceiling of an endless room stretching to infinity without walls. Vertex heights 2 and 3 are positioned midway between floor and ceiling. Water will pool in a three-dimensional P3 tiling and level off at vertex heights 2 and 3 before draining away. This may be seen in Figure 3b. The endless room contains both concave and convex versions of every motif and patch in a P3 tiling. But you can't tell which is which because they look identical in two-dimensions.

3D pavers can be created that have the same crenellated perimeters and configuration of rhombs as the original 2D pavers. They project an identical pattern in two-dimensions. But in order to lay a smoothly connected sculptural surface a second pair of 3D pavers are necessary to accommodate both concave and convex configurations. This 'negative' pair of pavers has been flipped inside out like a glove made to fit the other hand. All vertices have switched position, sliding either up or down a vertical axis.

Art as Research

Three-dimensional P3 tiling assembles intuitively in rigid materials as long as you stay within the floor and ceiling of the imaginary room. If your golden rhombs are cut accurately to size, local matching rules are not necessary. Figures 3a and 4b show sculptures made from stained glass and mirrored glass. These allow me to observe light projection and reflection. The skewed blocks of wood set with magnets shown in Figure 4c are helping me visualize rhombs as individual particles in a quasicrystal. Processes like laser-cutting, vacuum-forming, 3D printing, and traditional mold-making and casting, allow me to produce pavers in a range of materials. Constructing P3 tilings by hand, and paying close attention to materials and process, consistently reveals new information about the geometry.



(a) (b) (c)

Figure 4: *a) A pair of 3D pavers in wood. b) Work-in-progress on three-dimensional P3 tiling in mirrored glass. c) Skewed rhombic blocks of wood that assemble with magnets to form a P3 tiling.*

Summary and Conclusions

Painting configurations of rhombs to look like drums was fanciful but illuminating. It opened a door to a new way of seeing Penrose's tiling by rhombs. I started out in *Flatland*; traveled through an imaginary space with color, shade, and perspective; and explored a mathematically precise geometric figure in the messy materials of the real world.

Acknowledgements

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