Conjunction-forms: Three-Circle Combinations

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Abstract

This paper introduces conjunction-forms, shapes comprised of three circles in distinct combinations of proximity, tangency, and intersection. By varying the diameters of the component circles, four discrete types of conjunction-forms produce four complete form-sets. The 200 distinct conjunction-forms are classified and illustrated in tables, and the new features and classes of forms in each form-set are examined. The author presents artworks that employ conjunction-forms and explains how artistic strategies reveal their order and structure.

Introduction

Much of my studio artwork is composed with forms produced by combinatoric methods. My primary purpose is to produce a set of forms whose members are at once similar, in that all have the same basic features, and different, in that each is a unique arrangement of those features. Also important is the completeness of the set, in that all forms allowable by the system are present and no form repeats another after reflection or rotation. These same qualities are intended to be recognized in the artworks through the principles of visual art (color, scale, position, alignment, etc.). Overall, my work as an artist is an investigation of the possibilities and limits of a distinctly visual apprehension of structure and completeness.

At all stages of the process, from the development of the forms to the production of the artworks, my concerns are primarily visual: Are the important features of the shapes visually identifiable? Are the variations of those features also recognizable by sight? Is each combination visually distinct from other combinations? Is it apparent by sight that no form is repeated, even after reflection or rotation? Is the completeness of the form-set visually discoverable? These and other perceptual considerations guide decisions throughout the development of the form-set and the artworks composed of those forms; they will also be important to our discussion below.

The most recent form-system, the subject of this paper, was inspired by astronomical conjunctions. In astronomy, *conjunction* refers to the proximities and juxtapositions of stars, planets, and moons as observed from a given perspective against the background of the night sky. In this paper, conjunction will mean something similar but with geometric forms on the plane—*conjunction-forms*. The forms of concern here are comprised of three circles and might be more precisely called *three-circle conjunction-forms*, but we shall use the shorter term throughout this paper. As often happens in developing form-sets, the number of conjunction-forms grew significantly as new, unanticipated possibilities emerged from the combinatoric process. What was anticipated to be one small set of about a dozen forms unfolded into four distinct sets of 200 forms. The four form-sets and their particular features and orders, as well as some initial related artworks, are examined below.

Equal-Diameter Circles (LLL Form-set)

Development of conjunction-forms began with the question: How many distinct ways can three circles be united to produce a set of different but related forms? It became clear that a three-circle form could be better understood by defining its two-circle relationships. Each three-circle conjunction-form has three discrete two-circle combinations, which serve as the means to generate the complete form-set.

In this system, there are three ways (*modes*) to combine two circles at a time: *proximity* (nearby but not touching), *tangency* (edge-to-edge contact), and *intersection* (overlapping of interiors). A note about terminology: although tangency is defined mathematically as intersection of a single point, for brevity in

this paper, *intersection* shall be limited to describing circles with overlapping interiors. Each of the above modes can also be described by the number of *contact-points* between two circles: 0 points (proximity), 1 point (tangency), and 2 points (intersection). Thus, any form can be described by the unique combination of contact-points for each pairing of component circles. For example, a form made of three circles in a row (labeled A,B,C, left to right) with no gaps between would be described as 1,1,0: A-B = 1 (tangent), B-C = 1 (tangent), C-A = 0 (proximate) (see Table 1, group D, combination 1,1,0).

The three modes of conjunction—proximity, tangency, intersection—are the combinatoric variables of the conjunction-form system. Taking these modes three at a time for each form (repetition of modes is allowed), there are seven *combination groups*: (1) proximity only; (2) tangency only; (3) intersection only; (4) proximity and tangency; (5) proximity and intersection; (6) tangency and intersection; and (7) proximity, tangency, and intersection. We need to distinguish between combination groups and the particular combinations *per se*: the former identifies which modes are being combined, and the latter identifies the specific number of each mode in a given combination, i.e., the specific combination of contact points (see paragraph above). For example, Form-set 1 shows two specific combinations in the *Proximity & Tangency* combination group (see Table 1, Group D); these two differ in the specific combinations of contact points (see 1,0,0 and 1,1,0 in the *Modes* column) which produce two distinct forms (see column 1.1). Three of the seven combination groups have two possible specific combinations (Table 1, Groups D, E, F), so there are ten specific combinations in this form-set (not including forms in the dotted-line boxes, which are discussed below). The seven combination groups will remain constant in all four form-sets to be discussed, but the number of specific combinations will grow as we proceed.

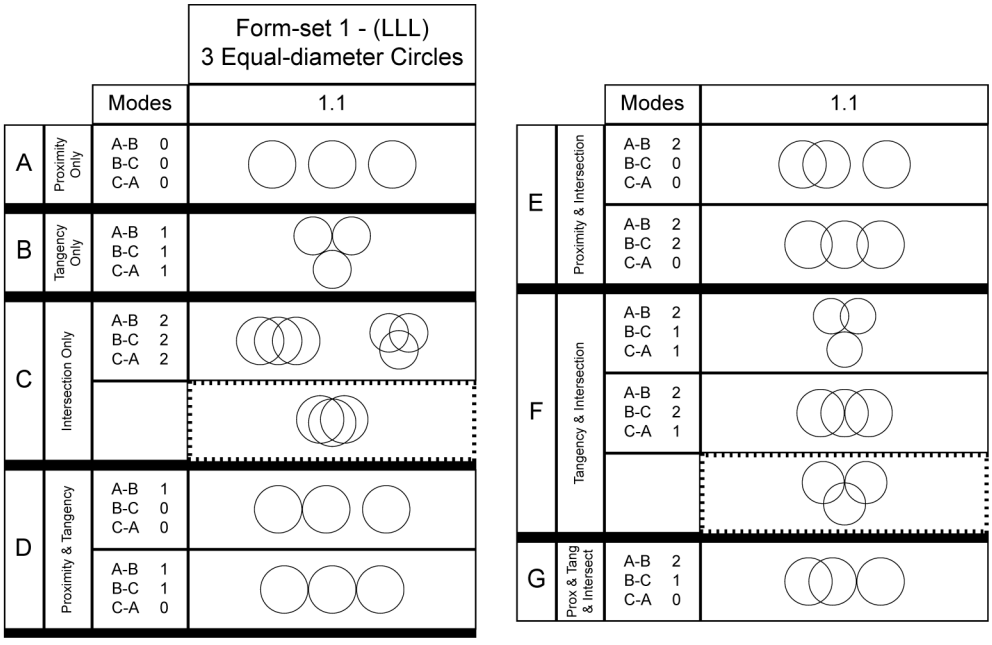

Table 1: *Form-set 1 (LLL).*

When making conjunction-forms the following conditions are adopted as constraints whenever possible: (1) circles are aligned on a horizontal central axis; (2) circles are spaced by standard distances (radius, half-radius, quarter-radius); (3) forms possess mirror symmetry on a horizontal axis; (4) forms that repeat another form after reflection or rotation are discarded from the set. If the aforementioned constraints are strictly followed, the equal-diameter form-set totals eight forms, leaving two of the ten combinations unrealized. The combinations 1,1,1 (tangency-only) and 2,1,1 (tangency & intersection) cannot be achieved with linear alignment (constraint-1 above), so a triangular alignment was adopted for these two combinations. With these additions, all ten combinations in the equal-diameter form-set have a

conjunction-form (see Table 1, solid-line boxes). Going forward, several other forms in the other sets will require exceptions to the above constraints, but most forms adhere to all constraints.

On examination of these ten forms, some further options emerged that would expand this set and point the way to new form-sets. First, my preference for visual clarity led to the addition of a triangularly aligned version of combination 2,2,2 which increases the distance between the contact points and enlarges the regions (Table 1, Group C, right-hand version). Second, some forms permit the coincidence of some contact points. Initially, it was assumed that the total points in a conjunction-form would be equal to the sum of the points of the modes, yet two of the combinations allow for a single point to assume more than one function. For example, the 2,2,2 intersection-only form permits joining three of the six intersection points into one point-location, resulting in a form that is visually distinct and possesses two fewer contact-points and one fewer region than the regular intersection-only forms (see Table 1, Group C, dotted-line box below solid-line box). A coincident-point form also occurs for the 2,2,1 form (see Table 1, Group F). Third, and most important, it became obvious even before the form-set was completed that no form could include a circle enclosing another circle. This circle-in-circle relationship could contribute much to artworks, so new combinations and new form-sets were undertaken and are the topics of the next sections. The equal-diameter form-set totals 13 forms: ten regular, one visual addition, and two coincident-point forms.

At this point, it will be useful to list the four conjunction-form sets and the shorter names by which we will refer to them in the remainder of this paper: (1) The equal-diameter set is called Form-set 1 or LLL (L for "large"—while not as precise as E for "equal," L is consistent with the other form-set labels). (2) Two larger circles and one smaller circle is Form-set 2 or LLS. (3) One larger circle and two smaller circles is Form-set 3 or LSS. (4) One large circle, one medium circle, and one small circle is Form-set 4 or LMS.

Two Larger Circles and One Smaller Circle (LLS Form-set)

In Form-set 2, one of the three circles is smaller in diameter than the other two. The size relationship between the smaller and larger circles is not fixed but relative. Although a ratio of 1:2 (the smaller circle is half the diameter of the larger) is adopted as a standard when possible, the smaller circle can increase or decrease in diameter if required to produce a given combination or for purposes of visual clarity. The important matter is that the two larger circles are exactly the same diameter to prevent either of the larger circles from enclosing the other (this relationship will be reserved for Form-set 4, LMS).

As with the LLL form-set (Table 1), the LLS form-set includes the seven combination groups (Table 2, Groups A through G) and a column (*class*) of forms without circle enclosures (Table 2, column 2.1). These forms are closely related to their equal-diameter counterparts (Table 1, column 1.1), but there are many more forms in this class than in Form-set 1. The LLL forms, comprised of a single kind of circle, differ by combinations only; permutations are not relevant in Form-set 1. By contrast, permutations impact the LLS forms in two ways: (1) The order of contact points among the circles in most combinations creates new forms. For example, in Group D the permutations 1,0,0; 0,1,0; and 0,0,1 create distinct forms (compare these to the single LLL form 1,0,0 in Table 1, Group D). (2) Permutations are also relevant to the spatial order of larger and smaller circles in the diagrammed forms. For example, in both 0,0,0 and 2,2,2, the two forms differ in the locations of the smaller circle in relation to the larger circles. Although spatial permutation produces only two additional forms here, it will add even more forms in the LSS and LMS sets. The key criterion for adding a new form in either permutation case (contact-points or spatial position) is symmetrical distinctness: if the new form does not repeat another after rotation or reflection, it is added to the set.

The most important and visually distinct additions to the LLS set can be seen in column 2.2. The inclusion of one smaller circle generates a new class of forms with the smaller circle fully enclosed by one of the larger circles. "Fully-enclosed" here includes tangencies on the inside of the larger, enclosing circle but excludes any intersection with the enclosing circle. There are 15 distinct members of this class of circle-enclosure forms. There are three instances where two distinct forms result from different permutations of spatial position (the boxes with two forms, Column 2.2, Groups E, F, and G). The right-hand examples in

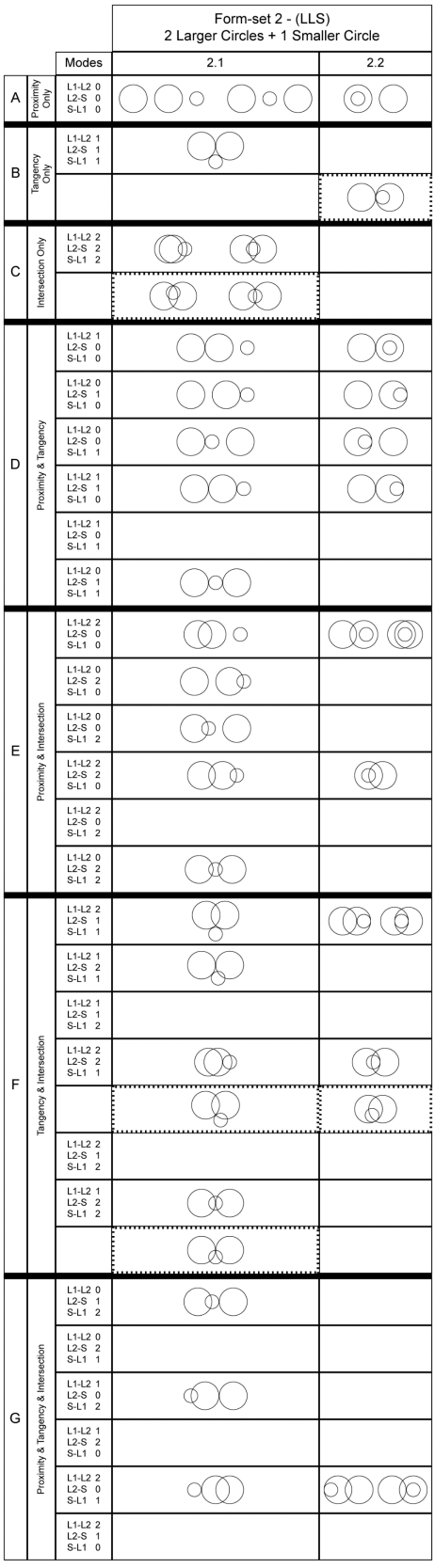

Table 2: *Form-set 2 (LLS).*

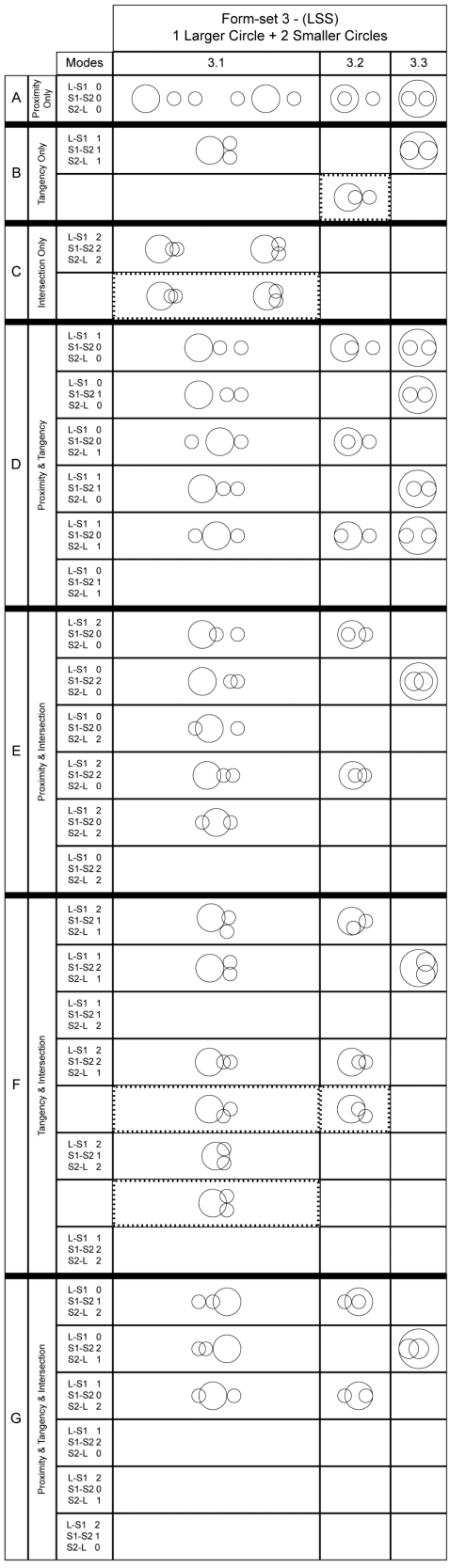

Table 3: *Form-set 3 (LSS).*

Group E, 2,0,0 and Group F, 2,1,1 are particularly interesting as the only forms in the set where the smaller circle is fully enclosed by both of the larger circles.

As seen in the LLL form-set, the LLS set also produces some coincident-point forms. These can be seen in the dotted-line boxes below the regular forms in the solid-line boxes in Groups B, C, and F, and these occur in both columns 2.1 and 2.2. Interestingly, the only means of creating the circle-enclosure form for combination 1,1,1 (Group B, column 2.2) is by a coincident-point form. This suggests a primary rather than a secondary role for coincident-point forms. Additionally, there are two asymmetrical forms in this set, both of which are coincident-point forms of 2,2,1 (Group F, dotted-line boxes). Asymmetrical forms are acknowledged here mostly for visual purposes; symmetry and asymmetry are particularly perceptible features that identify and differentiate forms in artworks.

Given the new forms arising in this set, it was clear that still more kinds of forms could be produced by varying the diameters further. Two more possibilities emerged: (1) One larger circle and two smaller circles would allow the enclosure of two smaller circles in the larger circle (*double-enclosure forms*). (2) Circles of three different diameters—one larger, one smaller, and one medial between the other two—would permit the special case of the small circle being enclosed by the mid-size circle, which in turn are enclosed in the large circle (*nested forms*), such as found in a target.

One Larger Circle and Two Smaller Circles (LSS Form-set)

The LSS form-set (Table 3) shares some close similarities with the previously-discussed LLS form-set (Table 2), including the same number of non-enclosed forms (compare columns 2.1 and 3.1, each with 26 forms). This is not surprising since both sets utilize circles with two different diameters, leading to similar permutations. Most of the LSS single-enclosure forms (Table 3, column 3.2) are also similar to their LLS counterparts (Table 2, column 2.2). However, Form-set 2 (LLS) produced 15 single-enclosure forms, while Form-set 3 (LSS) produced 12; we are missing the second regular versions of 2,0,0 and 2,1,1 (Table 2, Groups E and F, respectively) and there is one fewer form in Group D. The LSS set also has four asymmetrical forms: two regular versions of 2,1,1 and two coincident-point versions of 2,2,1 (Table 3, Group F, columns 2.1 and 2.2).

It is clear, however, that the salient contribution of this form-set is the new class of nine double-enclosure forms (Table 3, column 3.3). To realize some of the combinations in this new class, it was necessary to make the smaller circles slightly smaller or larger than the standard size of half the diameter of the larger circle. The essential constant is that the two smaller circles in any given form have exactly the same diameters to prevent either of the smaller circles from enclosing the other. Interestingly, there are no asymmetrical forms nor are there any coincident-point forms in the double-enclosure forms. And, of course, no forms in this class can include an intersection with the larger enclosing circle.

Large, Medium, and Small Circles (LMS Form-set)

The LMS form-set (Table 4) is the largest set with 99 forms, due in large part to the cumulative effect of producing its own versions of the previous three classes (columns 4.1, 4.2, and 4.3). Because Form-set 4 utilizes three different circle diameters, there are more distinct forms produced by both contact-point permutations and spatial-location permutations. For example, there are 44 non-enclosure forms (Table 4, column 4.1) compared to the 26 forms in each of Form-sets 2 and 3 (Table 2, column 2.1 and Table 3, column 3.1). Form-set 4 now includes a form for every contact-point permutation (i.e., every row has a form) and adds ten new forms with distinct spatial permutations (see rows with more than a single form in column 4.1). Many of the forms in columns 4.2 and 4.3 are recognizably related to their counterparts in Tables 2 and 3, yet there are also new versions of the single- and double-enclosure forms because more permutations are afforded by three kinds of circle. There are 39 single-enclosure forms (column 4.2) and eleven double-enclosure forms (column 4.3), both of which show more distinct forms than either of the previous form-sets (Table 2, column 2.2; Table 3, columns 3.2 and 3.3).

The new and final class of conjunction-forms is the nested set of three unequal-diameter circles. By "nested" is meant that the small circle is enclosed by the mid-sized circle, which are enclosed by the large circle. Again, "enclosed" includes both the proximate and tangent modes but excludes intersection; this means that no nested form can include an intersection. This limits this new class to only five forms, which fall into only those combination groups that do not involve intersection (Table 4, Groups A, B, and D).

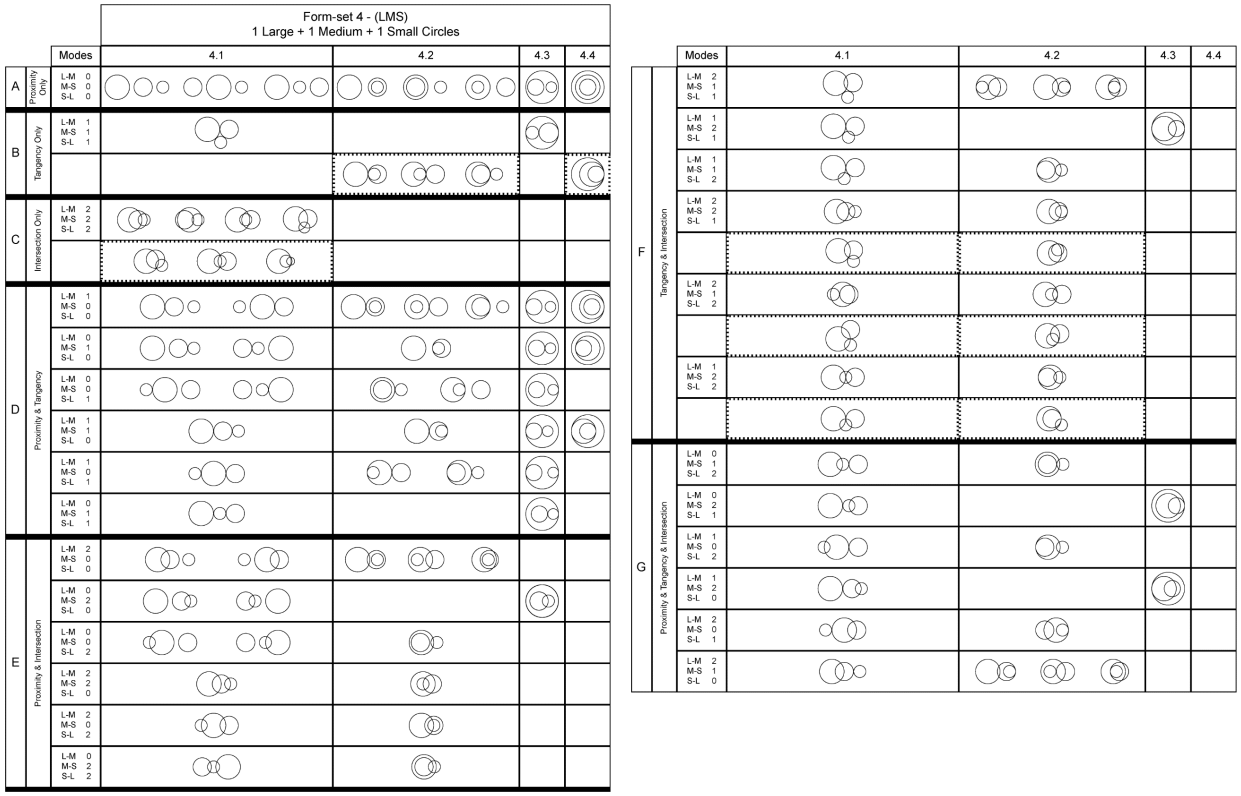

Table 4: *Form-set 4 (LMS).*

These, then, comprise the conjunction-form sets for use in artworks. A somewhat similar study by Haran and Sloane [1] examines all possible overlaps of three circles, but the authors employ different criteria than those used for conjunction-forms: 0-point proximity, 2-point overlap, and enclosure are allowed, but tangency and coincident points are not, and there are no limitations on diameters or alignment.

From Form-sets to Artworks

Among the purposes of the artworks is to make the structural aspects of the forms and form-sets, many of which have been discussed above, visually comprehensible. This includes not only an awareness of the similarities and differences among the forms, but also the completeness of the group of forms seen in any given artwork. Both of these tasks are linked, because to understand that all possible forms are present, the viewer should be able to recognize the constants and variables by which the forms are made—even to actively imagine form variations and seek them in the artwork. I also intend that this be accomplished by a purely visual encounter with the artwork, without the aid of verbal or mathematical description. The means to these ends lie in the precise employment of the perceptual principles of color, size, focus, position, alignment, proximity, angular orientation, etc., which constitute the formal language of art.

Artworks begin with a selection of forms, which typically means choosing a smaller subset from the larger form-set. I try to limit selections to no more than about 24 total forms; more than this number of forms in an artwork can often make it too difficult for the eye to recognize the aforementioned distinctness

of each form and completeness of the set in an artwork. Perhaps more important, the selected forms are typically a coherent subset. For example, selecting all forms within the tangency-only group (Group B of any of the form-sets) would rule out intersection or proximity as a variable, permitting the viewer to examine whether all possible tangent-only variations are present. So, each subset of forms is not selected arbitrarily but within certain combinatorial parameters—a "contextual completeness." Conjunction-forms offer two advantages over my previously-developed form-sets: First, the equal-diameter form-set (Form-set 1, LLL) is quite small and can be used in its entirety in a single artwork. Second, the four form-sets discussed in this paper are closely related by their mutual combination groups, making it possible to select subsets across the four form-sets.

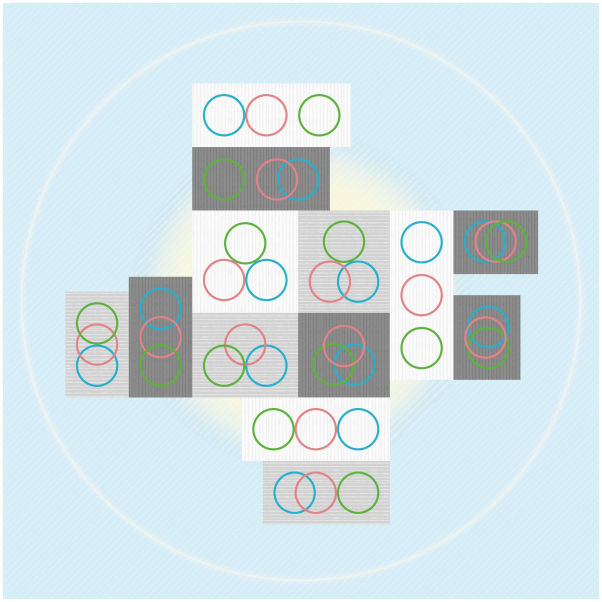

Figure 1: *"Surya" 2020.*

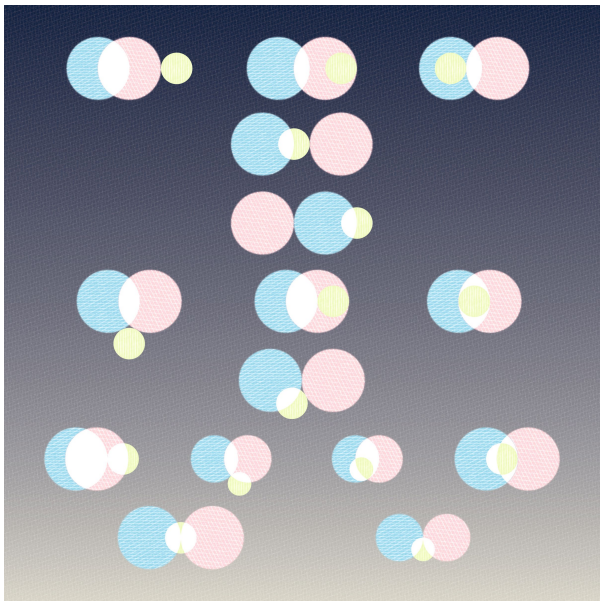

Figure 2: *"Transits" 2020.*

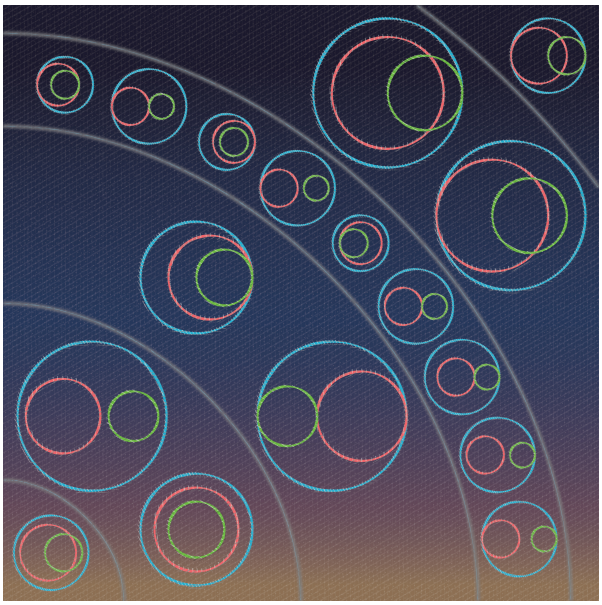

Figure 3: *"Horizon" 2020.*

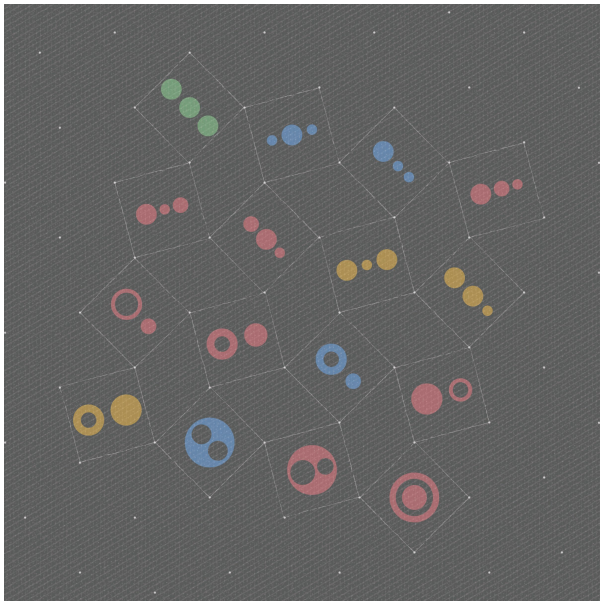

Figure 4: *"Clusters" 2020.*

The four artworks above show four discrete strategies for selecting forms: (1) *All forms in a form-set*. *Surya* (Figure 1) includes all 13 LLL forms (Table 1). (2) *All forms in one or more rows in a single form-set*. *Transits* (Figure 2) includes all 15 LLS forms in Groups F (ten tangency & intersection forms) and G (five proximity & tangency & intersection forms) (Table 2, Groups F and G). (3) *All forms in one or more columns in a single form-set*. *Horizon* (Figure 3) shows the subset of 17 double-enclosure and nested forms from the LMS set (Table 4, columns 4.3 and 4.4). (4) *Forms in rows or columns from multiple form-sets*. *Clusters* (Figure 4) includes all 16 forms in the proximity-only group (Group A) across all four form-sets: one from LLL, three from LLS, four from LSS, and eight from LMS (Group A in Tables 1, 2, 3, and 4).

The forms in these compositions are grouped, divided, aligned, sized, and colored to reveal their similarities and differences. For example, Figure 1 shows LLL forms (Table 1) on rectangles of three different shades of gray: white has all forms without intersections (Groups A, B, and D); middle gray has all intersecting forms with tangencies (Groups F and G); dark gray has all intersecting forms without tangencies (Groups C and E). These rectangles are distributed in four "arms" radially from the center while avoiding any edge-sharing by the same color. In Figure 2, the forms are grouped and differentiated by position and scale: the seven LLS (Table 2) permutations in Groups F and G (i.e., all forms that have both intersection and tangency) are aligned in seven horizonal tiers, and scale differentiates regular forms (larger) from coincident-point forms (smaller). In Figure 3, each of the six arcs hold LMS double-enclosure and/or nested forms (columns 4.3 and 4.4) from a single combination group; from lower left to upper right: one from group E, two from Group A, two from Group B, nine from Group D, two from Group G, and one from Group F. In Figure 4 the four colors indicate the form-sets to which each individual form belongs. The sixteen forms are, generally, distributed from "lighter" (non-enclosure) forms in the upper rows to "heavier" (enclosure and nested) forms in the lower rows.

Astronomy was not only the starting point for the development of conjunction-forms, but it persisted to the final stages as the subject-matter in the artworks. Celestial subjects are not specifically illustrated but rather generally evoked by color, alignment, and grouping. The artworks are intended as meditations on the multi-layered orders of the cosmos where, in its vastness, it seems that all possibilities of those orders may come to pass.

Conclusion

The close relationship among the four form-sets discussed in this paper is a new development in my work. The form-sets themselves can be understood as a combinatorial superset—each form-set taken individually is the complete set of mode combinations of three circles whose diameter relationships are constant, while the four form-sets taken together are the complete set of combinations of the three circle-diameters. These different combinatorial levels might be thought of as a "combinatorial harmonics." The artwork in Figure 4, its elements drawn from across the four form-sets, not only shows us all possible proximity-only conjunctions with three circles, but also shows these conjunctions in all four combinations of the three standard circle-diameters (indicated by colors). To be sure, these multiple levels of order challenge our perception, but I am intent on pursuing the possibilities of a visual comprehension of compound order.

The three-circle conjunction-forms will continue to produce artworks in future months and perhaps years. As important, the methods used to develop them are being adapted to produce new kinds of conjunction-forms with triangles, squares, and hexagons. And, even more intriguing from a visual standpoint, conjunctions of more than one species of shape (e.g., circles and triangles) are creating new and surprising region-shapes from intersections and tangencies. Much new work lies ahead, and the three-circle conjunction-forms discussed above are pointing the way.

References

[1] B. Haran and N. J. A. Sloane. "How Many Ways Can Circles Overlap?" *Numberphile*, video, April, 2019. https://www.numberphile.com/videos/overlapping-circles.