From Computer to Compass: Analysis and Reconstruction of a Self-Similar Islamic Geometric Pattern at Madrassa Madar-i-Shah

Phil Webster

Phil Webster Design, Athens, GA, USA; phil@philwebsterdesign.com

Abstract

This paper describes the re-creation of an especially elegant two-level Islamic geometric pattern found at the Madrassa Madar-i-Shah in Isfahan, Iran. The pattern is first understood using a "bottom up" analysis using computer-aided design software, followed by a "top down" reconstruction using traditional compass and straightedge techniques. In addition to the analysis and reconstruction itself, the paper considers the relative pros and cons of computer-aided design vs. traditional construction techniques when studying and drawing Islamic geometric patterns.

Introduction and Terminology

Figure 1 shows the beautiful mosaic tiling discussed here, along with magnified details of two portions of the tiling. A hi-res publicly available image can be found online at patterninislamicart.com [4]).

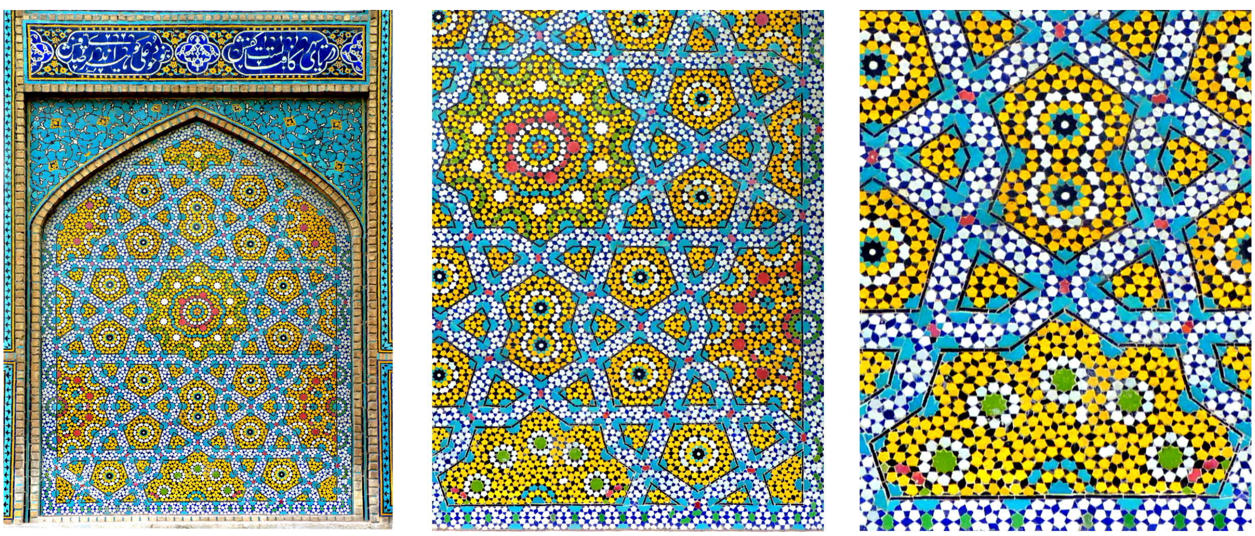


Figure 1: *The full archway tiling (left); lower right corner (middle); bottom center close up (right)*

Though relatively few in number, self-similar patterns are in many ways the culmination of the historical Islamic geometric idiom. While no definitive sources exist to explain exactly how such patterns were created, it is instructive to study their structure using a variety of approaches, both as a way to better understand historical examples, as well as to discover techniques for deriving new patterns using the same principles.

Jay Bonner [1, 2] has offered thorough expositions of the various types of self-similar ***Islamic Geometric Patterns (IGP)*** that exist in the historical record. The IGP discussed here is what he labels a ***Type C pattern***, where a larger scale pattern has had its lines widened, and then a smaller pattern fills both the tiles and the widened lines of the larger pattern (see Figure 2). In this paper I will call the larger pattern the ***first level***

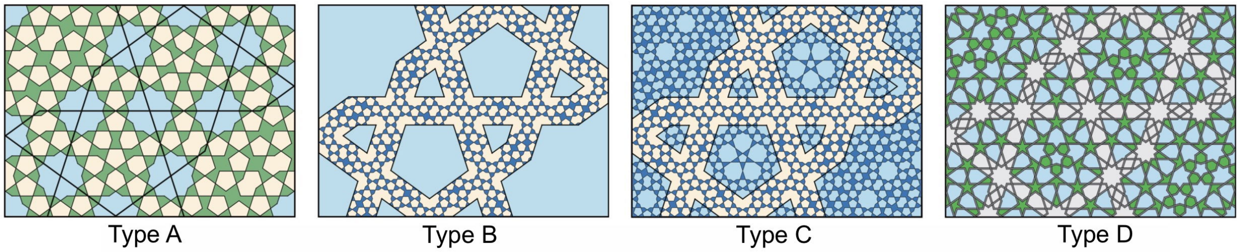


Figure 2: *Four historical types of two-level self-similar IGP (adapted from Bonner [2], with permission)*

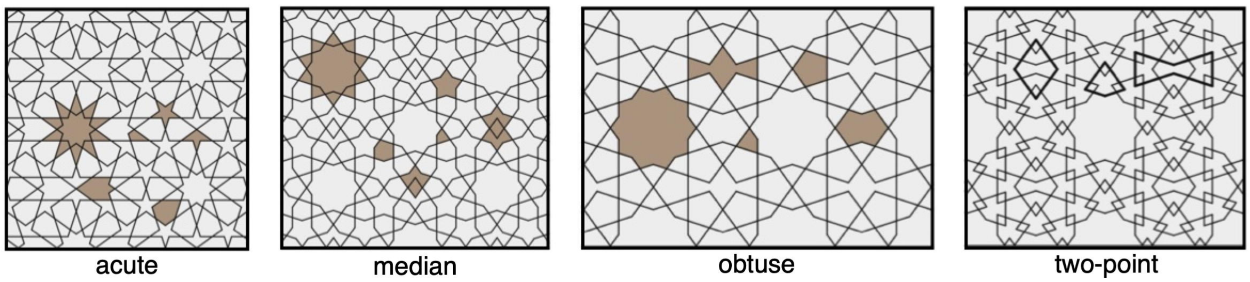


Figure 3: *Four historical IGP families (adapted from Bonner [2], with permission)*

pattern, and the smaller pattern the ***second level pattern***. I have also adopted Bonner's terminology for the four families of IGP based on the angles of the pattern lines relative to the underlying polygon grid; of the four families (see Figure 3), both the first and second level patterns in this case are of the ***obtuse*** variety.

Analysis of the Pattern Using Computer-Aided Design Software

I performed my initial analysis using the 3D modeling software SketchUp. The analysis was "bottom up" in the sense that I focused first on the smaller, second level pattern, using its proportions to establish the necessary size and band width of the larger, top-level pattern.

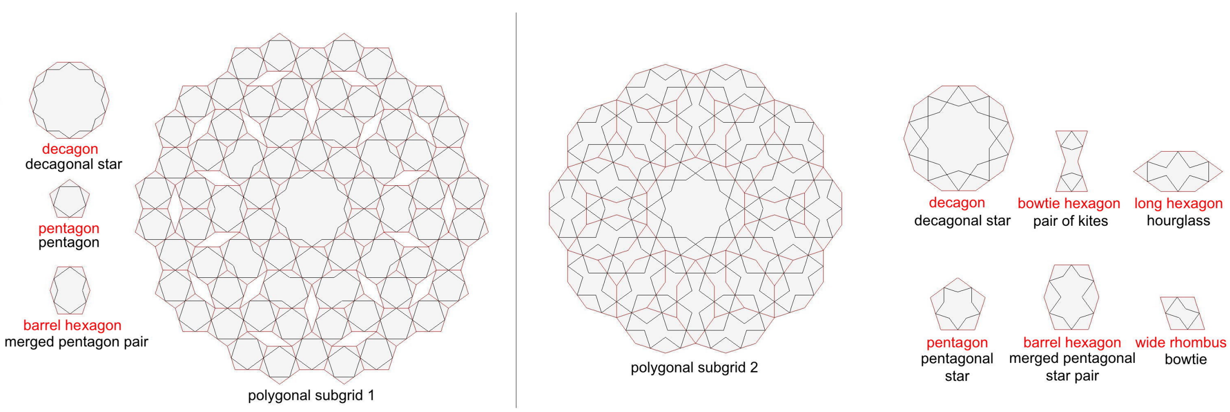


Figure 4: *Two possible polygonal subgrids for the second level pattern, with nomenclature for polygons and pattern shapes*

Structure of the Second Level Pattern

In Figure 1 (right) we can make out the component shapes of the second level pattern, which are easily recognizable as members of the 5-fold obtuse family (see Figure 3). As is often the case, there is more than one way to view these shapes as the result of applying pattern lines to underlying polygonal grids. Figure 4 shows two equally valid ways of deriving a portion of the second level pattern. On the left, a grid of central decagon, pentagons, and barrel hexagons are filled with lines emanating from edge midpoints and separated by 108 degrees. In this formulation, the polygonal grid directly generates decagonal stars, pentagons, and merged pentagon pairs; kites and hourglass shapes arise between them. On the right, a grid of central decagon, long hexagons, and bowtie hexagons are filled with lines emanating from edge midpoints and separated by 72 degrees. These three shapes yield decagonal stars surrounded by kites, hourglass shapes, and pairs of kites, respectively, with pentagons and merged pentagon pairs arising in between.

Ultimately polygon subgrid 2 proved easier to work with. The three shapes in Figure 4 (upper right) are part of a well known family, referred to by some as “girih” tiles [3], some others of which are shown (lower right). Surprisingly, the entire second level pattern can be recreated using just the top subset of three blocks.

Paying attention to the shapes of the first level pattern, we see that there are three different edge lengths present (see Figure 5). The kites have 2 short and 2 medium length edges; the merged pentagonal pairs have 4 medium and 4 long edges; the pentagons have 5 long edges; and the decagonal stars have 20 medium edges. Critically, each short, medium, and long edge has exactly the same arrangement of second level shapes (and hence underlying polygons) as every other (Figure 5, left). All three first level edge types have second level decagons with their centers positioned at the end points and their edge midpoints lying on the first level edges. Along short edges the decagons meet directly edge to edge; along medium edges, one bowtie hexagon lies lengthwise between the decagons; and along long edges two bowtie hexagons lie lengthwise between the decagons. A little trigonometry shows that the three edge lengths have relative lengths of $\Phi : \Phi + 1 : \Phi + 2$ (where Φ is the golden ratio, $\approx 1.618...$). This conformity of the edge arrangements allows all of the first level tiles and band segments to blend seamlessly wherever they meet along a common edge length.

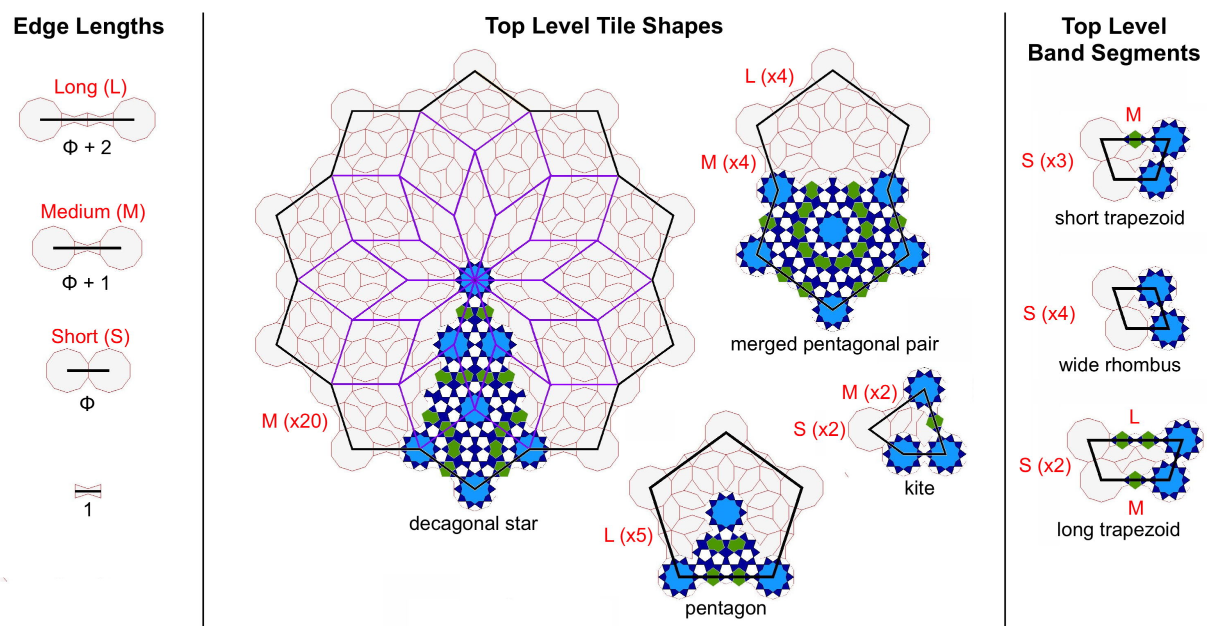


Figure 5: *First level tiles and band segments with edge lengths, subgrid polygon arrangements, and portions of derived second level pattern*

Once the edges of each polygon are established, the centers are easily filled with more copies of the same three polygons (Figure 5, center). In fact, numerous such arrangements are often possible; generally, those selected emphasize the rotational or mirror symmetry of the first level polygon. In the case of the decagonal stars, an “ideal” arrangement can be created using a decomposition of the star into the skinny and wide rhombi native to the 5-fold system—shown by purple lines—with each rhombus having its own mirror and rotational symmetry. Filling the subgrid with pattern lines as shown in Figure 4 yields the final pattern.

Establishing the First Level Band Width

A similar procedure can be applied to the widened lines between the first level shapes, which inherently establishes their width relative to the shapes in between. These bands comprise three types of segments—wide rhombi, short trapezoids, and long trapezoids—which again have the same three edge types with the same second level polygon arrangements, and are also easily filled with more copies of the same second level subgrid polygons (Figure 5, right).

Structure of the First Level Pattern

Part of the beauty of this tiling is that the first level comprises exactly the same set of shapes as the second level —namely, decagonal stars, kites, pentagons, and merged pentagonal pairs. This pattern can be recovered in its original form simply by drawing lines down the centers of the widened bands (see Figure 6).

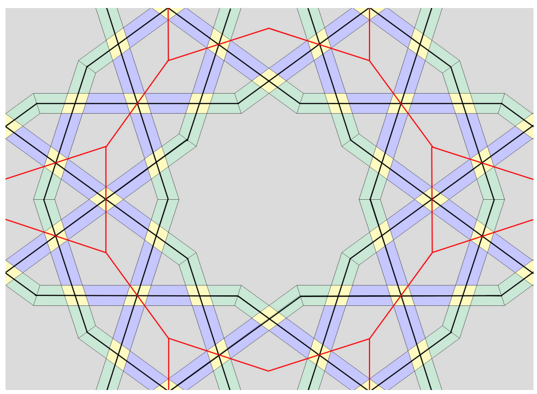


Figure 6: *First level base pattern (bold black lines) with its own subgrid (red lines), widened bands (colored segments) and reduced tile areas (gray)*

Reconstructing the Pattern Using Compass and Straightedge

Having gained a good understanding of the pattern, I set about recreating it using the traditional tools of compass and straightedge. This is necessarily a “top down” approach—as it must have been for the original designers and builders—since what is given is the overall available dimension (be it a wall or a sheet of paper) and the pattern must be fit into the existing space. By convention in this tradition, the pattern cannot just be arbitrarily trimmed; instead, to maintain the aesthetic, the pattern must be constructed so that the centers and lines of symmetry of the pattern coincide exactly with the border.

Laying Out the Grid

The first level pattern has decagon centers lying on a wide rhombic grid. Figure 7 illustrates the construction of this grid. In short, the large circle with diameter AOB is divided into ten points 1-10 using a standard decagon construction. These points establish the 36° angles needed to extend lines and create the rhombic

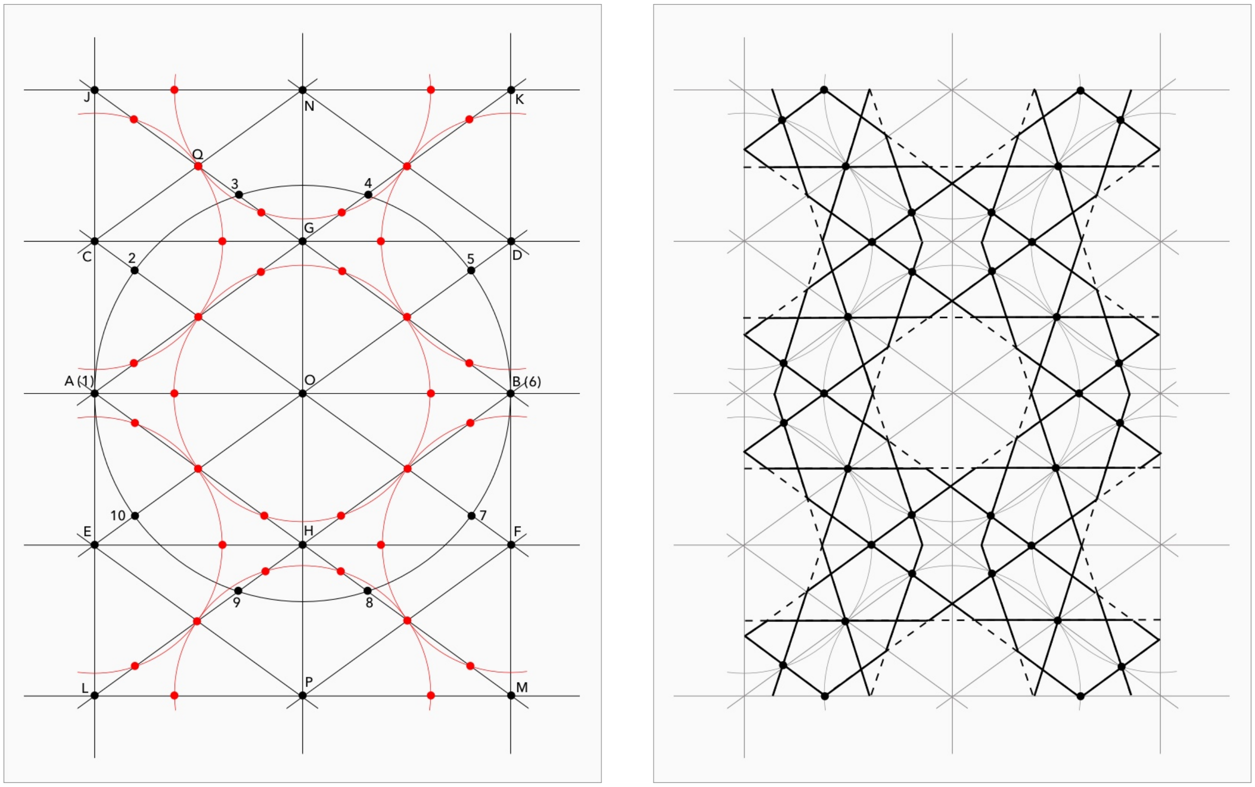


Figure 7: *Diagram for the manual construction of the first level pattern. Creating the rhombic grid and key points (left); connecting points to form the first level pattern (right)*

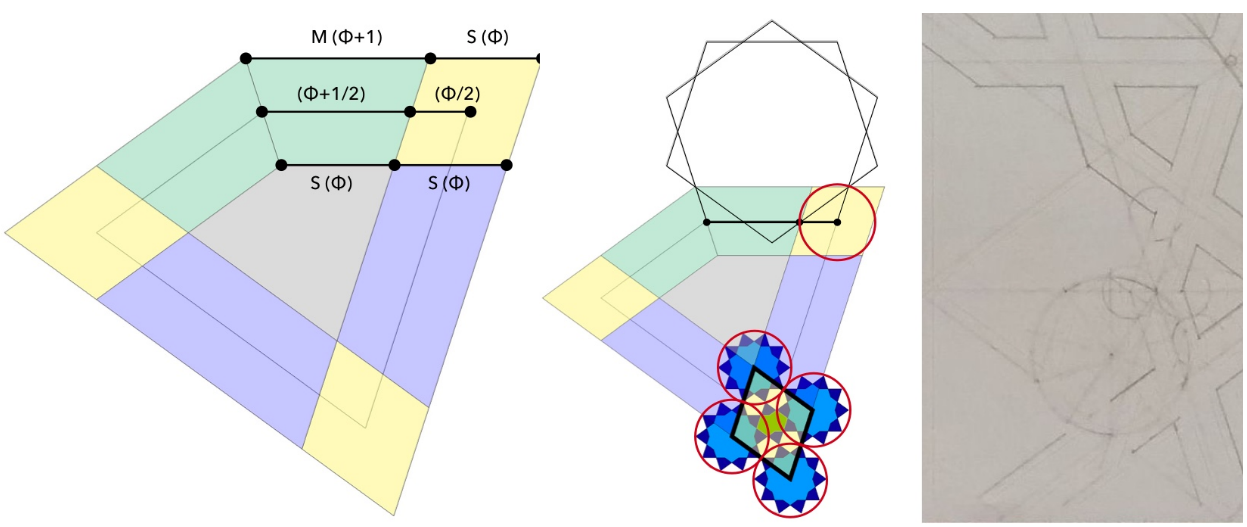
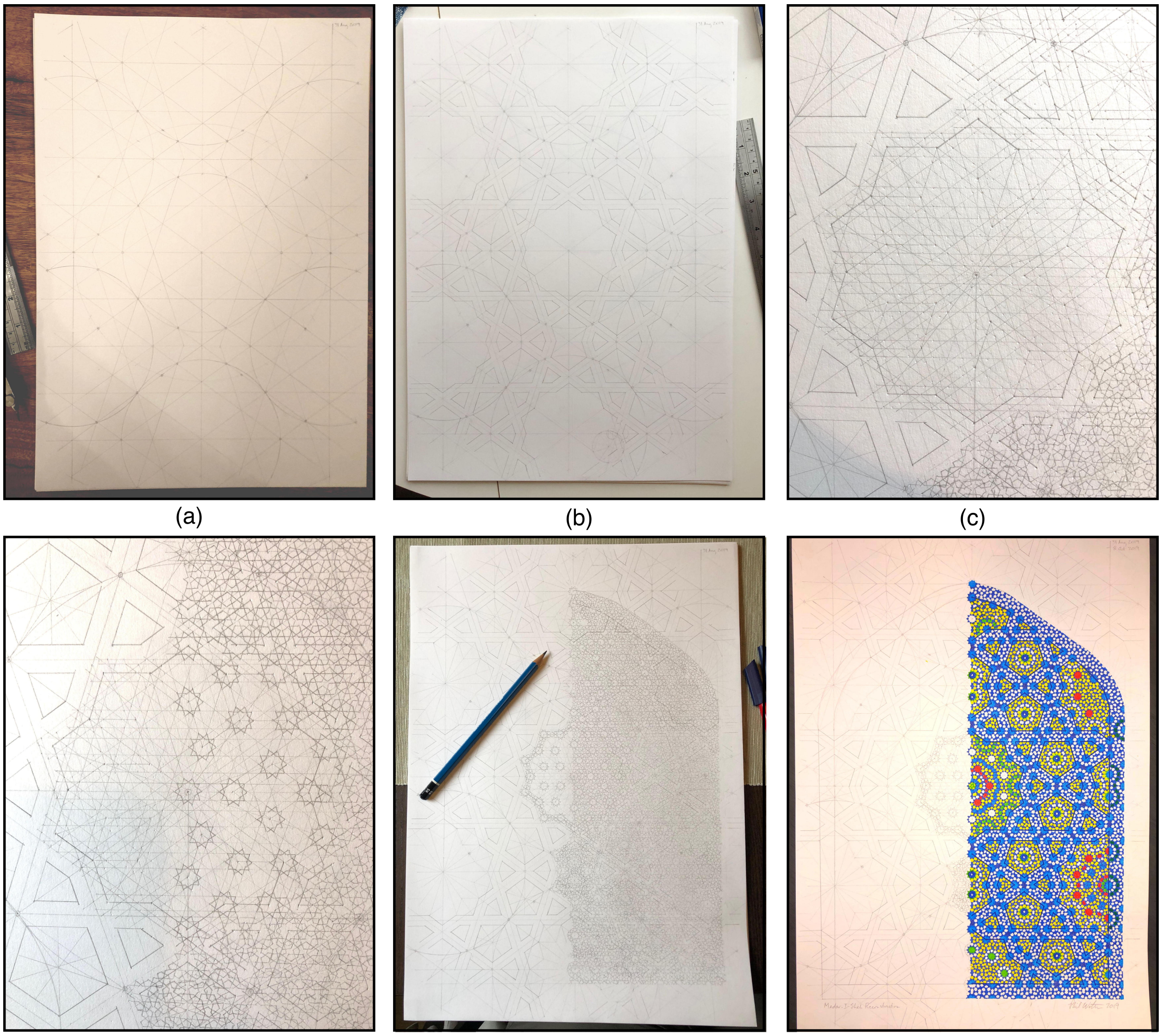


Figure 8: *Constructing the proper band width. First level kite with proportions (left); a construction that yields this proportion (center); the actual construction in practice (right)*

grid. Interested readers can find a step-by-step description of this construction at philwebsterdesign.com [5].

The points C, D, E, F, N, O, and P in Figure 7 are the centers of the subgrid's decagons, and NQ is the inradius of those decagons. To do a full construction we would normally proceed to construct the subgrid of decagons and bowtie hexagons as in Figure 6 (red lines), and then find all of the edge midpoints and construct the pattern from there. However, this proves unnecessary in this case, because all of the midpoints we need are already present, namely, all the places where the inradius circles intersect the horizontal and diagonal grid lines (see Figure 7 (left), red points). Figure 7 (right) shows how these points can be connected and extended to form the first level pattern. Figure 9a shows this first stage of the actual artwork.



(a) (b) (c)

(d) (e) (f)

Figure 9: *Photos of manual construction: (a) Construction of top level pattern (per Figure 7); (b) Widening of bands (per Figure 8); (c) 10-fold star points placed with divider points, and lines of alignment; (d) 10-fold stars drawn freehand; (e) Completed second level pattern; (f) Final artwork.*

Widening the First Level Bands

Having established the first level pattern, the lines now need to be widened to create the bands. Focusing on the kite shape in Figure 6, and using the edge length proportions from Figure 5, one can calculate that the short side of the kite needs to be divided by proportion $\Phi + 1/2 : \Phi/2$ to establish the proper band width (Figure 8, left). Luckily, this happens to be the proportion of the side of a decagonal star to the total span between alternate points (Figure 8, center). So, I used this construction to establish the proper division of the kite edge, which equates to the radius for the second level pattern decagons (Figure 8, right). Figure 9b shows this second stage of the actual artwork.

Drawing the Second Level Pattern

Having established the first level grid with bands and the radius for the second level decagons, it would be theoretically possible to construct an entire subgrid, and then all of the shapes for the second level pattern. However, this would be impractical and require far more effort than is needed for a successful outcome. In order to make the drawing of the second level as efficient (yet as accurate) as possible, I used several methods (see Figure 9c and d). (1) Rather than draw circles for every decagonal star, instead I used dividers and marked the star points with small holes. This is more accurate as the divider tip is smaller than the width of a pencil lead. (2) To determine the 10 directions for these holes, I used surrounding pattern lines as guides and placed them by eye. At this small scale, it was possible to achieve accurate-enough placement in this manner. (3) I filled in the decagonal stars first, drawing the edges freehand, as placing a straightedge for such short lines would have been overly time-consuming. (4) As more stars were marked, alignments across several stars became clear. I marked these lightly in pencil, and they served as cross-checks for alignment of the other shapes between the decagonal stars.

Anomalies of Color and Symmetry

Several choices in the tiling notably break symmetries of the “ideal” version of the pattern. In the absence of evidence, we can only speculate as to why. Given the sophistication of the design, it seems highly doubtful that these anomalies were accidental on the part of the designer. I find it plausible that some were artistically motivated, while others are likely errors made by less knowledgeable (or careless) workers.

Referring back to Figure 1, the different coloring schemes within the top level decagonal stars and in the side borders are clearly intentional. In the central star, a potential ring of 10-fold second level stars is replaced by a ring of red shapes with overall 5-fold symmetry. This ring also “points” to the right rather than the top, breaking vertical mirror symmetry. In the lower corners, rings of potential 10-fold second level stars have been replaced by clusters of three blue tiles. In the side borders, arcs of teal-colored tiles add texture and form to an otherwise more uniform area.

Variations that are probably errors include two stray white (and one missing blue) tile in the bottom central star area, and various differences between the left and right borders—especially the disconnected teal “snake” in the lower right (vs. the one at lower left), and the one stray purple tile in the upper right.

A much more interesting and subtle “error” in symmetry involves the short and long trapezoid band segments throughout the pattern. While these are necessarily asymmetric internally (see Figure 5), they could have been arranged with rotational symmetry around top level pentagons and decagonal stars, and mirror symmetry around top level pentagonal pairs—but in most cases, they were not.

Why Use Traditional Tools?

Having seen the immense amount of time and effort required to construct the pattern using traditional tools, one might naturally ask: “Why bother? Why not just use the computer?” It’s a question I asked myself a few times while working on the piece! After all, working with the computer offers many obvious benefits: perfect precision; rapid experimentation; and easy copying of numerous pattern elements, to name a few. However, I believe working with traditional tools provides several benefits of a completely different nature.

1. Deeper Appreciation of Historical Craft

When creating (or re-creating) a pattern on the computer, handling large numbers of objects, and operations such as copying, mirroring, or performing an N-fold rotation, are trivially easy and nearly instantaneous. Given this facility, it is easy to forget just how skilled and dedicated the artisans of centuries ago must have been to design and execute patterns of this complexity. Producing a work by hand gives one a deep appreciation of the effort involved in creating these patterns without the aid of a computer.

2. Additional Insights into the Pattern

During the many hours of drawing the second level design, I came to appreciate certain aspects of it that were not as apparent during the computer reconstruction. The alignments of the arms of the decagonal stars, whose lines then coincided with parts of the outlines of other shapes, was one. Another was the many small variations in color and symmetry employed by the original artisan to create visual variety, only some of which are discussed in the previous section.

3. Slower Pace and Meditative State

Working by hand requires time and focus. By slowing the pace, entering deep focus, and carrying out many repetitive actions, I would enter a pleasant, almost meditative state very different from what I experience when working on a computer.

4. Artistic Satisfaction

Lastly, as satisfying as it is to generate a beautiful pattern on a computer screen, there is an entirely different level of satisfaction in creating something with one's own hands. The very time and effort required, which may from one point of view seem inefficient and unnecessary given today's tools, are the very thing that imbues the piece with additional meaning and artistic satisfaction.

Summary and Conclusions

This paper shares a detailed analysis and reconstruction of the basic two-level pattern in this historical example. Certain interesting nuances, such as the exact shape of the curved arch, and additional anomalies in color and symmetry from the theoretical ideal, have unfortunately been omitted due to space constraints.

The process of understanding this pattern through two very different means illuminated not only the pattern itself but the value of working with traditional tools and techniques. I hope other mathematical artists are inspired to step away from their computers from time to time and use their hands to create some of their art, and that they experience some of the satisfaction I have found in doing so.

References

[1] J. Bonner. "Three Traditions of Self-Similarity in Fourteenth and Fifteenth Century Islamic Geometric Ornament", *Bridges Conference Proceedings*, Granada, Spain. August 1-2, 2003, pp. 1-12. http://archive.bridgesmathart.org/2003/bridges2003-1.html

[2] J. Bonner. *Islamic Geometric Patterns: Their Historical Development and Traditional Design Methodology*. Springer, 2017.

[3] P. Lu and P. Steinhardt. "Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture", *Science*, 315:1106, 2007.

[4] D. Wade. *IRA 1118 : Madrassa-yi-Madar-i-Shah, Isfahan*. https://patterninislamicart.com/archive/main/3/iran/ira1118.

[5] P. Webster. *Phil Webster Design - Resources*. http://www.philwebsterdesign.com/resources.