3D Aperiodic Girih Tiles

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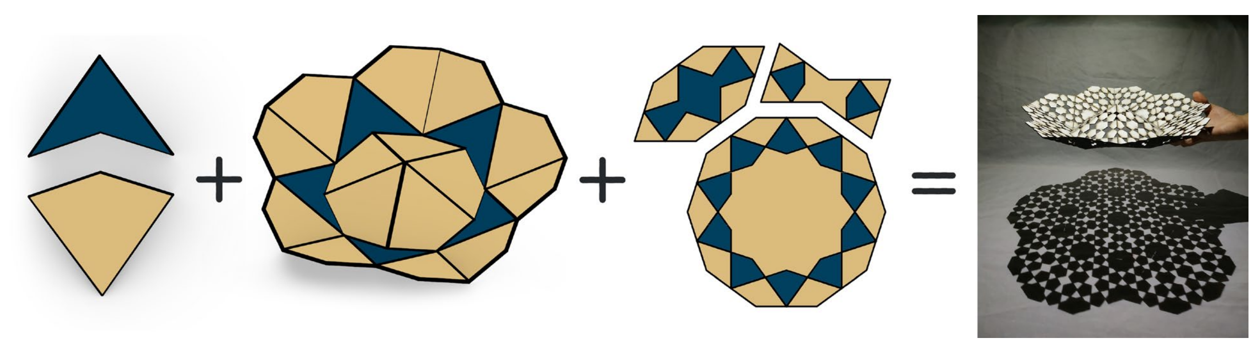
Abstract

The purpose of this paper is to develop the Islamic geometric patterns from planar coordinates to three dimensions with aperiodic symmetry. We are proposing techniques that cover four families of girih: acute, median, obtuse, and two-point tessellation by using the historical method, polygons in contact (PIC), in our pattern production. What's more, we employ Penrose P2 and P3 prototiles as our main template. This study effectively answers the question regarding the gap between planar design from medieval Islamic architecture and contemporary demands in modern art and design.

Introduction

The studies of Islamic art and ornament has shown that the Islamic artisans had a vision for developing their design consistency. They developed their design from simple cross-star (grid) patterns to sophisticated dual-level design. However, there is no unanimity among researchers about the existence of self-similar and/or quasicrystalline structures in the historical examples. Therefore, we are interested in generating three-dimensional aperiodic designs both to develop Islamic geometric patterns in a higher dimensions and to create their planar aperiodic projections (Figure 1).



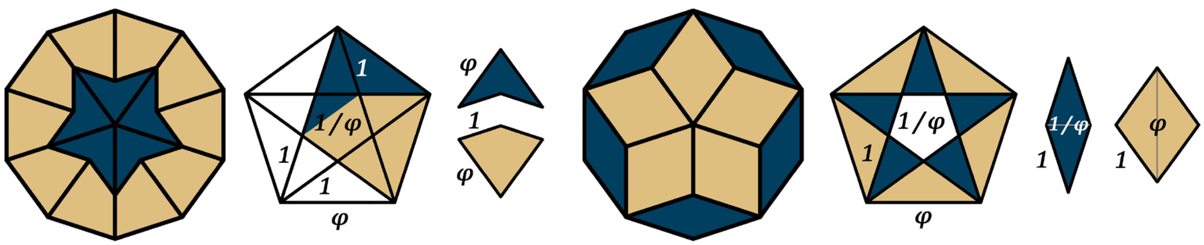
(a) (b) (c) (d)

Figure 1: *The process of our investigation to create 3D aperiodic girih: (a) Penrose P2, kite and dart prototiles, (b) orthogonal projection on vertices of P2 tile that results in 3D Penrose tiling, (c) traditional girih tiles, and (d) the output of our studies*

Covering a surface with 5, 8, 10, and 12 axes of symmetry with only translational symmetry is impossible and results in gaps in the tiling. Aperiodic tessellation was discovered by mathematicians in the early 1960s and after twenty years, made its way to the study of natural quasicrystals. Dan Shechtman discovered an alloy, aluminum-manganese, in 1984, which had unusual diffractograms that today are known as revelatory of quasicrystal structures [11]. Due to the importance of aperiodic tiling, we will describe Penrose prototiles and their lattice projection tilings in the first section. In the next section, we show the way to associate a girih pattern with Penrose tiling. In the last part, we are demonstrating the results of Penrose-type tiling in 3D.

3D Penrose Tiling

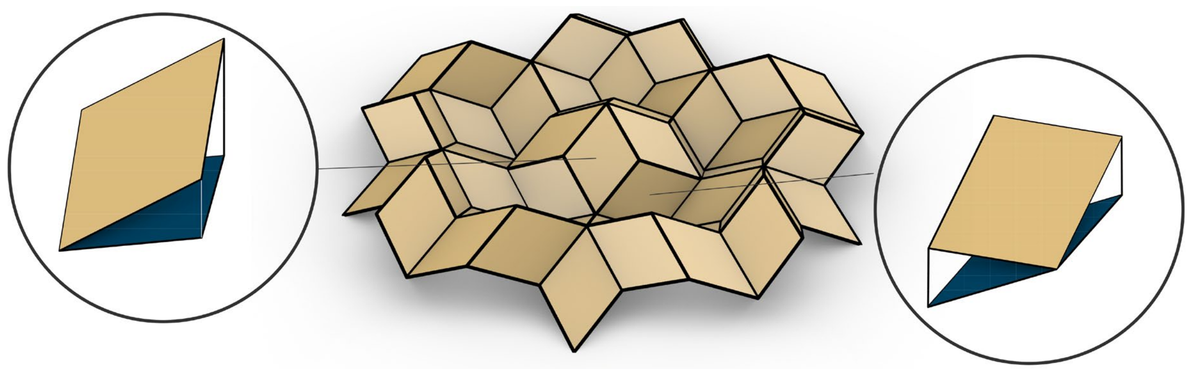
The beginning of the application of subdivision and inflation to build a quasiperiodic tessellation, which excludes translational symmetry, came to prominence in the 1970s. The Penrose kite and dart, which was the result of an investigation in aperiodic sets of tiles, was the most famous example. Roger Penrose proposed aperiodic tiling under three types and named them P1, P2, and P3 [6]. Furthermore, those types can create an aperiodic tiling through three separate methods, including self-similar subdivision, tiles with matching rules, and projection of a slice of a cubic lattice in $\mathbb{R}^5$ [2]. Penrose tilings feature a finite number of shapes, known as prototiles, which can tile the plane with no gaps or overlaps. Penrose prototiles also have many common features derived from the pentagon, and consequently, the golden ratio. The technical details and matching rule proofs are described by Martin Gardner [5] (Figure 2). Penrose P1 has six prototiles that create more complexity in both mathematics and geometry. For this reason, we exclude this set of tiles from our method.



(a) (b)

Figure 2: *Penrose prototiles and their relation to the pentagon (golden ratio): (a) the Kite and Dart tiling (P2) and (b) the rhombus tiling (P3).*

De Bruijn [3] devised a method based on the pentagrid tiling to construct an infinite aperiodic plane without the use of inflation and substitution. This technique has very striking results which one of that states by raising the vertices (by a particular method of indexing the vertices) of the thin and thick rhombi in three dimensions, all of the rhombi can be forced to be congruent, and results in 3D Penrose tiling [3, 6]. Later, Duneau and Katz systematized this method and proved that the 3D Penrose tiling can be obtained by a projection from a 6D unit cubic lattice. In general, they proved that we can project any point in tilings from p-dimensional space $\mathbb{R}^n$ as the projection, from a higher-dimensional space $\mathbb{R}^m$, where $m > n$ [4].



(a) (b)

Figure 3: *Three dimensional Penrose patterns or as de Bruijn called them, Wieringa roofs [3], in the hope that some enterprising architect would use it for the ceiling of a large room [6]: (a) the orthogonal projection on the horizontal plane for the thick rhombus and (b) the orthogonal projection on the horizontal plane for the thin rhombus*

The orthogonal projection on the horizontal plane is a thick rhombus, and the short horizontal diagonal of the space rhombus is projected as the short diagonal of the thick rhombus. Likewise, the orthogonal projection is a thin rhombus, and the long horizontal diagonal of the space rhombus is projected as the long diagonal of the thin rhombus. These orthogonal tiles are called golden rhombi, because the ratio the long diameter to the short diameter is $1{:}\,\varphi$ (1:1.618033987….), which is the golden ratio. The golden rhombus is our base template for mapping girih patterns in three-dimensional aperiodic tiling. Now the question is how to appropriately fill the golden rhombus with girih tiles?

Geometric Transformation

Rigby [10] introduced the idea of generating aperiodic girih patterns using kite and dart prototiles and called them Penrose-type Islamic patterns. Therefore, the concept of generating planar Islamic aperiodic patterns is not new. Likewise, Bonner and Kaplan [1] illustrated Penrose rhombi prototypes with the polygons in contact (PIC) method. We used the PIC method for creating girih tiles, which were first introduced by Hankin [8] based on his observations in a historical area in India. This method has been developed and analyzed by many researchers throughout the years. Jay Bonner [1] established his method based on the Hankin method and investigated different methodologies for reaching the optimal way to create girih patterns. Consequently, we are using his method for creating our designs in P2 and P3 sets of tiles. We chose ten-pointed stars as the *fivefold* system.

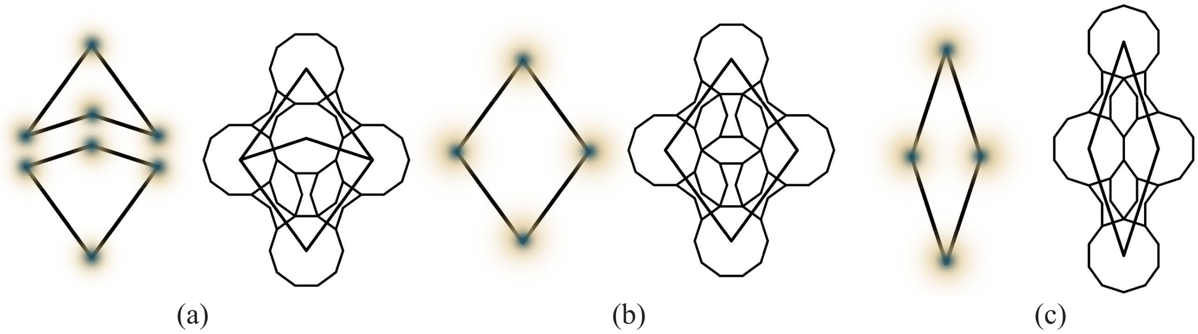


Figure 4: *The polygonal arrangements in P2 and P3 prototiles in: (a) kite and dart, (b) thick rhombus, (c) thin rhombus*

As a result, we centered a decagon at each vertex of the prototiles to begin our girih tessellation (Figure 4). The scale factor between rhombi and the side of the girih tiles, define the polygonal arrangements for the populating the rhombi. This scale factor is an expression of the inherent proportions of the generative system. In our case, the scale factor is $2 \times \left[\sqrt{(\phi+1)^2 - (\phi/2)^2}\right] \approx 1{:}\,1.498$, which has the Figure 4 arrangement.

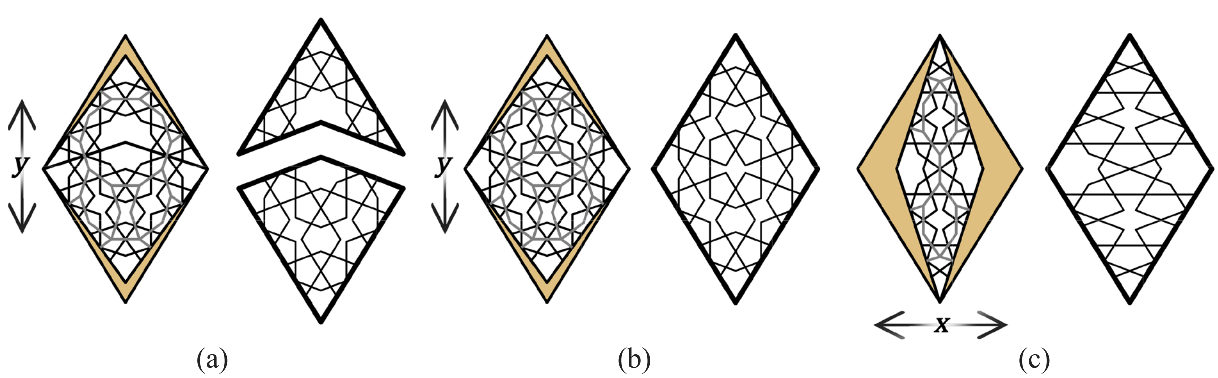


Figure 5: *Geometric transformation in P2 and P3 prototiles: (a) kite and dart stretch with the invariant of the x-axis, (b) thick rhombus stretch with the invariant of the x-axis, and (c) thin rhombus stretch with the invariant of the y-axis*

We use the ratios between the golden rhombi and the Penrose prototiles to define a geometric transformation mapping the girih patterns onto our templates. It can be obtained with either the stretch transformation or one-dimensional scaling with just one axis depending on the type of prototile. For the thin rhombus, the stretch factor (k) is 1.90 with the invariant y-axis (Eq. 1). Also, due to the fact that P3 prototiles are corresponding to the P2 prototiles, the thick and the congruent kite and dart have the same ratio. For this reason, they have the same stretch factor, 1.17, but this time the x-axis is invariant (Eq. 2). The following matrices map the vertices of the original P2 and P3 to the golden rhombus.

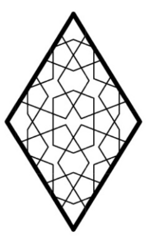
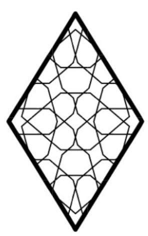
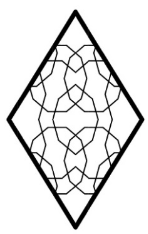
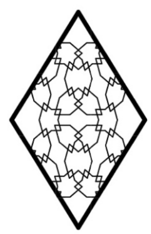
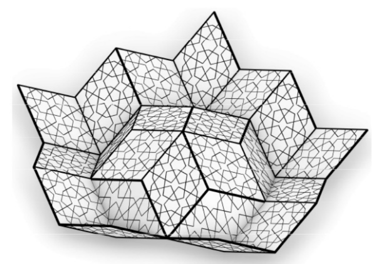
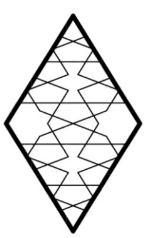
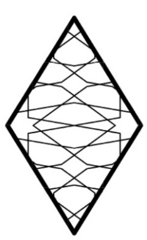
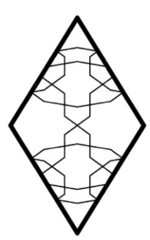
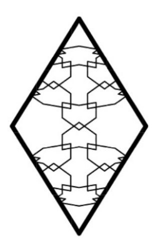
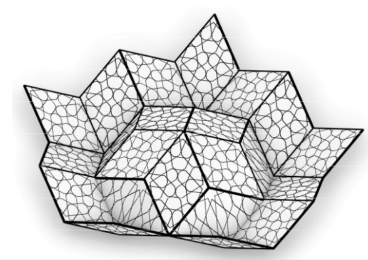
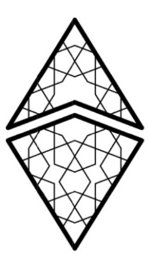
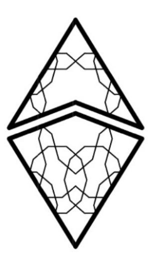
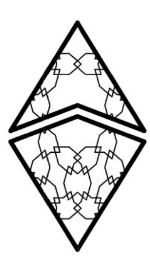
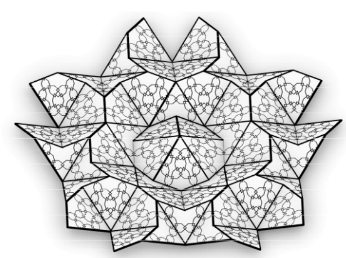
$$\begin{bmatrix} 1.90 & 0 \\ 0 & 1 \end{bmatrix} \quad (1) \qquad\qquad \begin{bmatrix} 1 & 0 \\ 0 & 1.17 \end{bmatrix} \quad (2)$$

As we can see in Figure 5, we have the girih tiles scaled to the golden rhombi and ready to tile the space. in the next section, we review different alternatives for creating 3D patterns.

3D Girih Tiling Alternatives

The underlying polygons which demonstrated in Figure 4 and 5 have the ability to create a wide range of patterns. In this section, we show the girih families arising from different conditions. These types of girih are categorized based on their angles on the underlying polygons. The angle between lines extracted from the midpoint represents each family. For the acute family, the angle is 36°; for the median family, the angle is 72°, and for the obtuse family, the angle, is 108°. Two-point tilings extract lines from two points with equal distance from each other. With all that said, we have P2 and P3 prototiles ready for tiling. These all gives us 16 different prototiles, and 8 different tiling in total, which is presented in Table 1.

Table 1: *The full range of pattern production with the four families of girih*

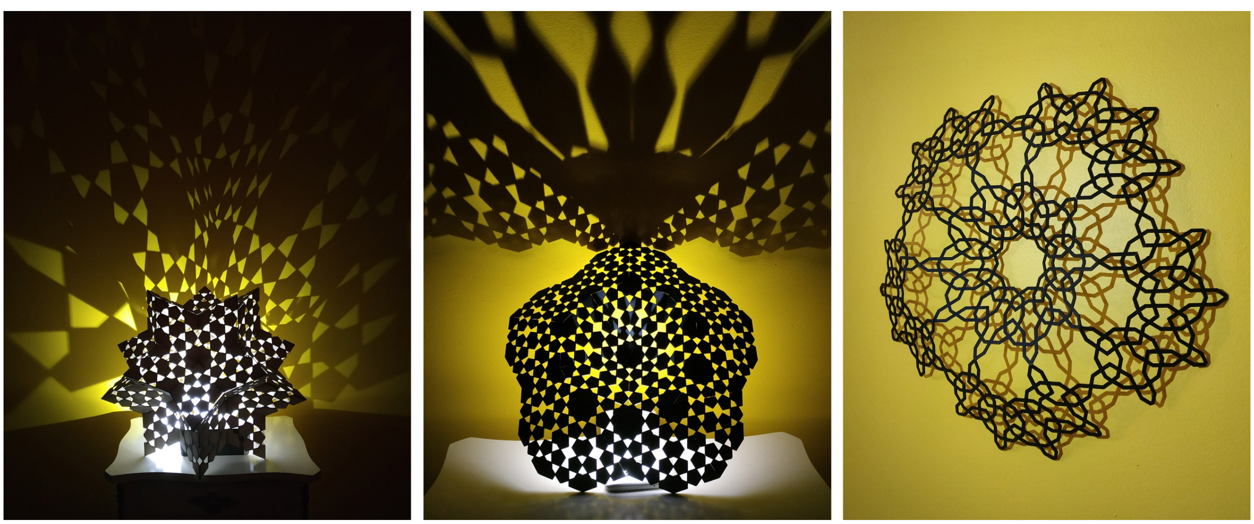
Tile	Obtuse	Acute	Median	Two-point	Example
Thick Rhombus					
Thin Rhombus					
Kite-Dart					

Making Some Girih Tiling!

Now it is time to use our algorithm to create 3D patterns with the help of computer-aided design and manufacturing (CAD/CAM). For the sake of prototyping, paper was preliminarily used as a material for generating tiles. By taping the edges of the tiles, we can complete the tessellation. The Grasshopper plugin was used to generate a precise model to optimize no gaps in different materials. Figure 6 shows the models that were produced by creating a script for the Girih pattern. A separate script was developed for each of the four families of girih tiles described in Table 1. Different materials were combined with different digital fabrication methods. The materials explored were: paper, cardboard, and plexiglass. Laser Cutting and CNC were our fabrication methods.

(a) (b) (c)

Figure 6: *Paper and cardboard models of Penrose-girih tiles: (a) aperiodic acute ten-pointed star, paper model, (b) aperiodic obtuse five-pointed star, (c) rhombic hexecontahedron model with obtuse ten-pointed star on the faces*



(a) (b) (c)

Figure 7: *Playing with light and shadow: (a) rhombic hexecontahedron, (b) ten-pointed star kite and dart pattern, (c) Median ten-pointed star kite and dart pattern with widened line*

Our goal was to produce physical models and prototypes for interactive use of 3D girih. In the first model, we constructed 3D P3 tiles as the surfaces that were laser cut and glued from paper. The flexibility of the paper allowed us to determine the possibilities of tiles with curved surfaces such as dome and squinches (Figure 6). In the next stage, we used cardboard, which has the potential to create void boundaries on the surface without deformation. This unique feature generates girih shadows that have the planar girih specifications, with the help of the beam of lights (Figure 7a).

Now, how about using our golden rhombus template for generating stellation? In geometry, stellation is the operation of creating polyhedra by increasing the facial planes past the polyhedron edges of a given polyhedron until they intersect. The rhombic hexecontahedron (RH) is among the stellations that have 60 golden rhombic faces with icosahedral symmetry, and they are self-supporting stellations of the rhombic triacontahedron. Also, in a 1987 article, Guyot [7] reported the appearance of the RH shape in quasicrystals. The relation of RH solids and quasicrystals form is not clear yet, and there is controversy among the scholars but, partial planar projection of RH reveals thin and thick Penrose prototiles. These all features inspired us to create a hollow self-support RH girih, demonstrated in Figure 6c. Sándor Kabai has done many studies in this area, and the stellations of RT are detailed and proofed in his article [9].

Summary and Future Work

We have shown an algorithm for generating 3D Penrose girih. There are certainly other methods for creating the 3D pattern; nevertheless, we have presented a high range variation of patterns. Further study and work is needed to establish complex symmetries in other sets of art like dual-level designs. As an example, we are working on optimizing these patterns for self-supporting structures, and eventually, they would work as roof coverage. We hope to report some retractable-girih structures in the near future.

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