Family Tree of Impossible Objects Created by Optical Illusions

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Abstract

This document classifies a family of "impossible objects" created by Kokichi Sugihara that are real physical objects but appear to be impossible due to optical illusions and organizes them into a family tree according to the reasons why they look impossible. There are two fundamental sources of ambiguity in reconstructing 3D objects from 2D images that allow us to create impossible objects. One is the lack of depth information in images. The other is the freedom in choosing the viewpoint from which we see the images. The proposed family tree will provide a new framework for further research on impossible objects and 3D optical illusions.

Introduction

Originally, the phrase "impossible objects" referred to imaginary 3D structures that can be represented by pictures, called "impossible figures," but cannot be realized as real physical objects. Oscar Reutersvärd drew many such figures and is known as the father of impossible figures [1]. Penrose and Penrose [4] designed simple impossible figures such as the "Penrose Triangle." These impossible figures were also depicted in woodcut prints by Escher [2]. These activities took place in the early- and mid-20th century.

Soon after, several tricks were found to construct real 3D objects whose appearances match the impossible figures. Figure 1 shows two typical tricks using (a) the Penrose triangle as an example. The first trick, shown in Figure 1(b), is a hidden-gap trick, in which the object is constructed with a discontinuity that appears connected when viewed from a certain angle [3]. The second, shown in Figure 1(c), is a curved-surface trick, in which the object is constructed with curved surfaces that look planar [1]. Both of them give the same appearance as the Penrose triangle when seen from specific viewpoints.

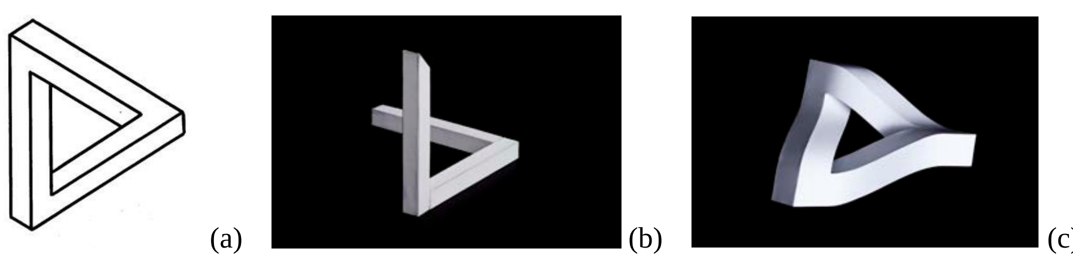


Figure 1: *Construction of 3D objects from a 2D impossible figure: (a) Penrose triangle; (b) object created by the hidden-gap trick; (c) object created by the curved-surface trick.*

Sugihara discovered another trick, which uses angles other than 90 degrees that appear rectangular [5]. It is thusly called the "non-rectangularity trick." He constructed many real 3D objects from impossible figures [6]. Additionally, he used this trick in a different way to create "impossible motion illusions," in which objects look ordinary, but certain motions associated with the object appear impossible [7].

Inspired by these findings, Sugihara created various types of real 3D objects whose appearance or behavior appear to be impossible. He used the term "impossible objects" to refer to these real objects. In the present paper, we classify Sugihara's impossible objects into 11 groups and attempt to place them within a tree structure, which we call a "family tree of impossible objects," according to the mechanism driving their illusion, their visual effect, and the method of their construction. First, we argue that there are two types of freedom in designing 3D structures from 2D images, and in the following two sections we present seven types of impossible objects stemming from the first type of freedom and four types stemming from the second. Finally, we construct a family tree of impossible objects to serve as a framework for understanding how these different objects are related.

Freedom in the Choice of 3D Structures from 2D images

As shown in Figure 2, given a fixed 2D picture and viewpoint, we try to reconstruct a 3D structure whose projection coincides with the picture. We cannot determine this 3D structure uniquely because the same 2D picture might be an image of, for example, a small object a short distance away, a large object a long distance away, a thick object, or a thin object. There is thus a large amount of freedom. Mathematically, we can construct a system of equations to represent the 3D structure represented by a given 2D picture. If the picture correctly represents a 3D object, the system has infinitely many solutions [5]. Let us call this ambiguity the *freedom in choosing the depth*.

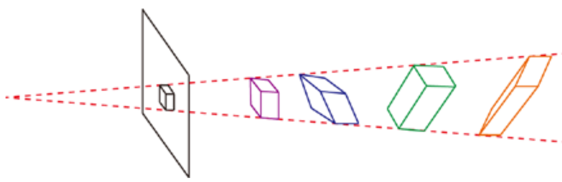


Figure 2: Freedom in choosing the depth.

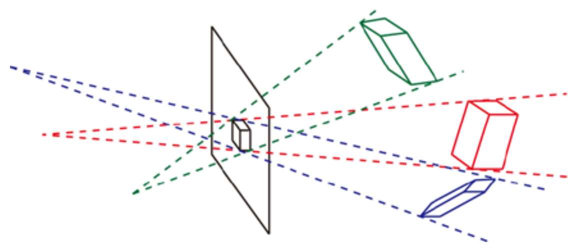


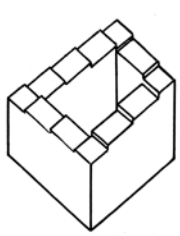
Figure 3: Freedom in choosing the viewpoint.

There is also another type of freedom. When we are given a 2D picture, we usually have no information about the angle from which we should view the picture. As shown in Figure 3, we can observe the picture from any viewpoint. However, the reconstructed set of possible objects (i.e., the solutions of the system of equations) changes when we change the viewpoint. Let us call this ambiguity the *freedom in choosing the viewpoint*.

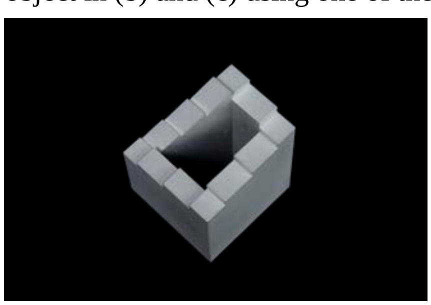
Impossible Objects Created Using Freedom of Depth

The object represented by a single picture is not unique. However, we usually perceive a unique object up to scale. This gap between the mathematical properties and the nature of human vision is one of the main reasons why impossible objects can exist.

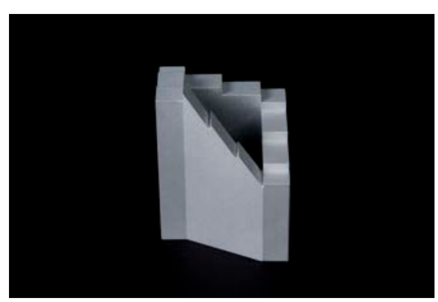
Anomalous Objects are 3D structures whose appearances match impossible figures when viewed from a certain angle [6]. An example is shown in Figure 4, which depicts (a) an impossible figure, namely, an "endless loop of staircases," (b) the 3D structure seen from the special viewpoint, and (c) the same 3D structure seen from a different angle. When we see the picture in (a), we usually feel it cannot be constructed as real 3D object. Indeed, if we imagine walking up the staircases counterclockwise, we eventually reach the starting point again, which is impossible. However, the system of equations associated with this picture has solutions, and we can construct a real 3D object from any of the solutions. We constructed one such object in (b) and (c) using one of these solutions.



(a)



(b)

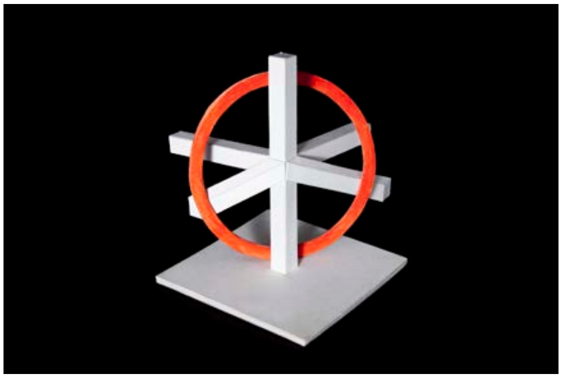


(c)

Figure 4: *Anomalous object: (a) impossible figure; (b) real 3D object; (c) another view of the object.*

The reason why we feel the figure in (a) is impossible is that we are apt to interpret the picture as a 3D object whose faces are connected by right angles. If we can connect the faces by arbitrary angles, we can construct the object as shown in (b) and (c).

Impossible motion objects are objects that seem ordinary when standing alone, but which appear impossible after the insertion of a second object [7]. Figure 5 shows an example, where (a) shows an ordinary object consisting of a vertical pole and four horizontal perches. However, the inserted red ring seems to be hanging in an inconsistent way; it passes behind the pole, but also passes in front of all four perches. Figure 5(b) shows the object from a different angle, from which we can understand that the perches are all extending backward, and hence the insertion of the ring is not impossible. The feeling of impossibility comes from the human preference for perceiving objects at right angles. When we see the scene in (a), we tend to interpret the four perches as extending horizontally at right angles to one another. Even though the ring is hung as shown in (a), it is difficult to guess the true orientation of the perches.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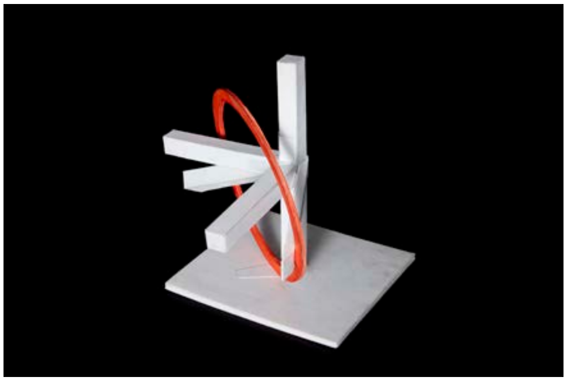
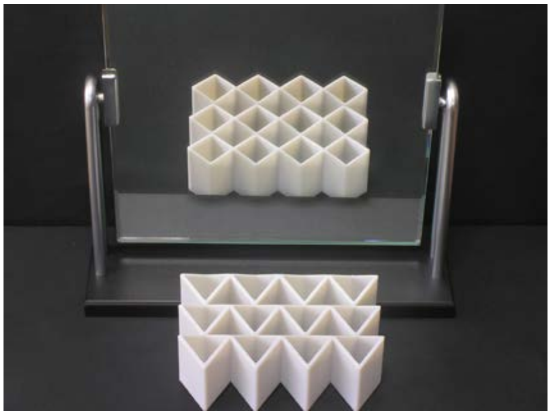

(b)

Figure 5: *Impossible motion object: (a) object with an inconsistently hanging ring; (b) another view.*

Ambiguous objects are objects whose appearances change drastically in a mirror. Examples are shown in Figure 6, where (a) represents the change from triangles to rectangles, (b) represents the change from diamonds to all card suits. In both cases, the direct appearances and the mirror reflections are quite different.

We can construct these objects by combining two systems of equations for two desired 2D shapes and solving them [8]. The solution represents a space curve. We construct a cylindrical object by moving the curve vertically and collecting all the points swept by the curve. The resulting object appears to be a cylinder with uniform height, and the top curve of the cylinder appears to be a planar curve perpendicular to the cylinder axis, even though it is not flat. In this sense, this class of impossible objects is also based on the freedom of depth and human preference for rectangles.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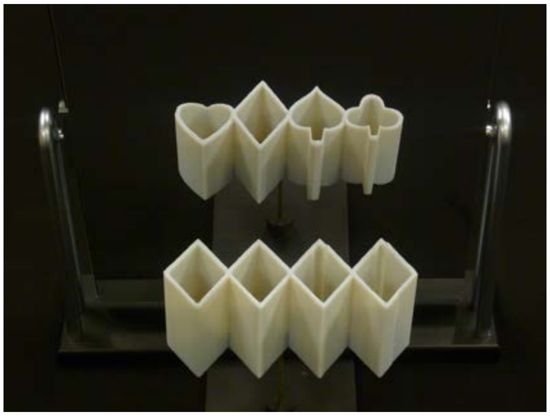

(b)

Figure 6: *Ambiguous objects: (a) triangles and rectangles; (b) card suits.*

Topology-disturbing objects are objects that change not only their appearance but also their topology when viewed in a mirror. Figure 7 shows an example: in (a) the two cylinders are separated in the direct view but are intersecting in the mirror; (b) shows the same object from a different angle. These objects can be constructed using the same principle as the ambiguous objects, namely, by first decomposing the second appearance into touching but non-intersecting parts, applying the method for ambiguous object construction to each decomposed pair, and finally adjusting their heights in such a way that the parts appear to be touching in the mirror [12]. The construction of this type of object takes advantage of the human preference for interpreting continuously connected parts as belonging to the same object.

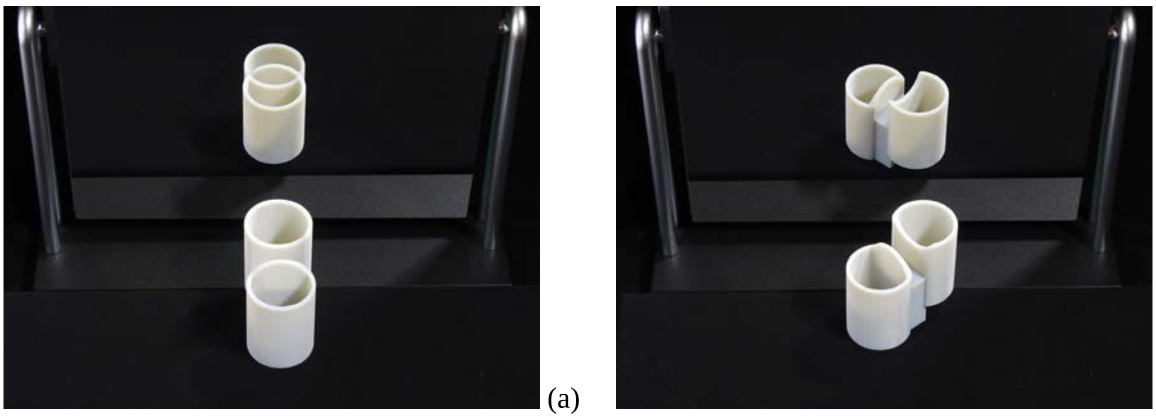

(a) (b)

Figure 7: *Topology-disturbing object: (a) separated cylinders that have an intersecting mirror reflection; (b) the same object from a different angle.*

Deformable objects are objects that appear to deform when we change our viewpoint continuously. As an example, Figure 8 shows a series of snapshots of an object during rotation around a vertical axis by 180 degrees from left to right. The object originally faces rightward, and after rotating by 180 degrees, it appears to face to the right again. This behavior of the object makes it seem as though it is being deformed, although it is made out of an inflexible material. This object can be constructed by applying the ambiguous object method for a pair of arrows -- one facing right, and the other facing left [11].

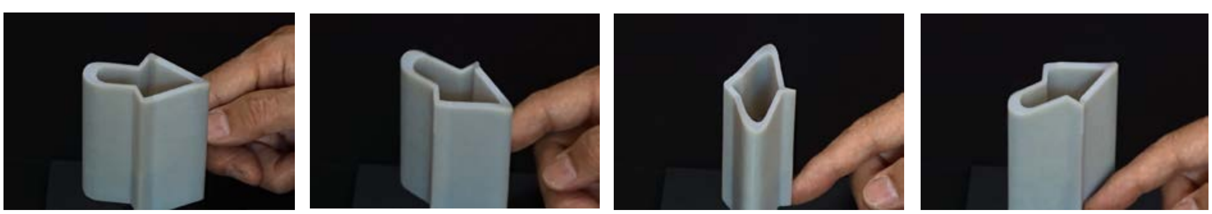


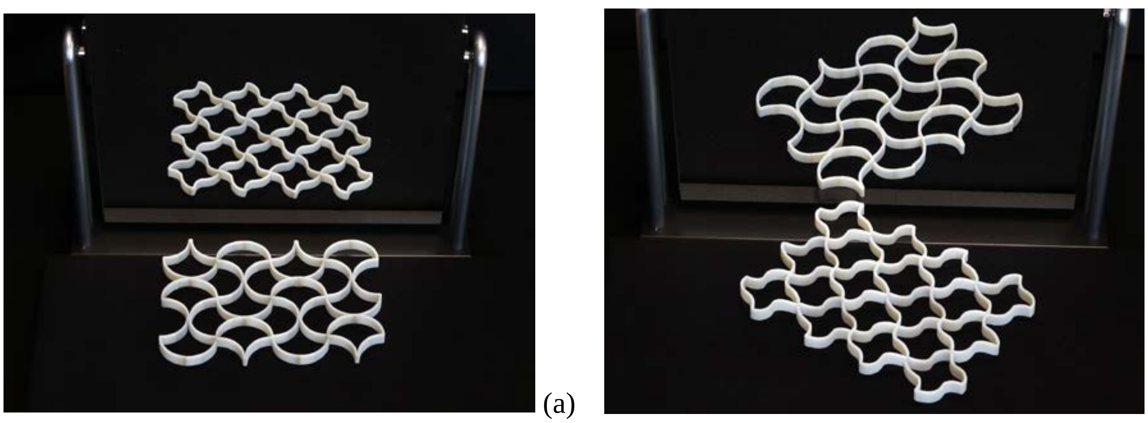
Figure 8: *Deformable object "An Arrow that Likes to Face Rightward."*

Reflexively-fused objects are objects that appear to be nonsense, but when placed on a mirror, the objects and their reflections together create meaningful shapes. An example is shown in Figure 9, where (a) shows an isolated object, (b) shows the object on a mirror, and (c) shows the object from a different angle. As seen in (b), the shapes of card suits are created despite not being mirror symmetric with respect to a horizontal line. We can construct these kinds of objects by first splitting a target shape by a horizontal line into the upper and lower parts, reversing the lower part so that it is upside down, and finally applying the ambiguous object method to both the upper part and the reversed lower part [13].

(a) (b) (c)

Figure 9: *Reflexively-fused object, "Card Suits Floating on a Mirror."*

Ambiguous tiling is a class of objects that give two tiling patterns when viewed from two different angles [15]. An example is shown in Figure 10. The direct view represents a tiling consisting of "gingko leaves" while the mirror image represents a tiling consisting of "maple leaves." Note that each tiling consists of the same tiles, and both cover the plane without gaps or overlaps. This class of objects can be constructed by finding a pair of tile patterns that satisfies two properties simultaneously. One is the tiling property wherein each tiling unit can be placed on the plane without overlaps or gaps, and the other is the ambiguous object property wherein the two patterns can be realized as two appearances of the same object.



(a) (b)

Figure 10: *Ambiguous tiling "Gingko Leaves and Maple Leaves."*

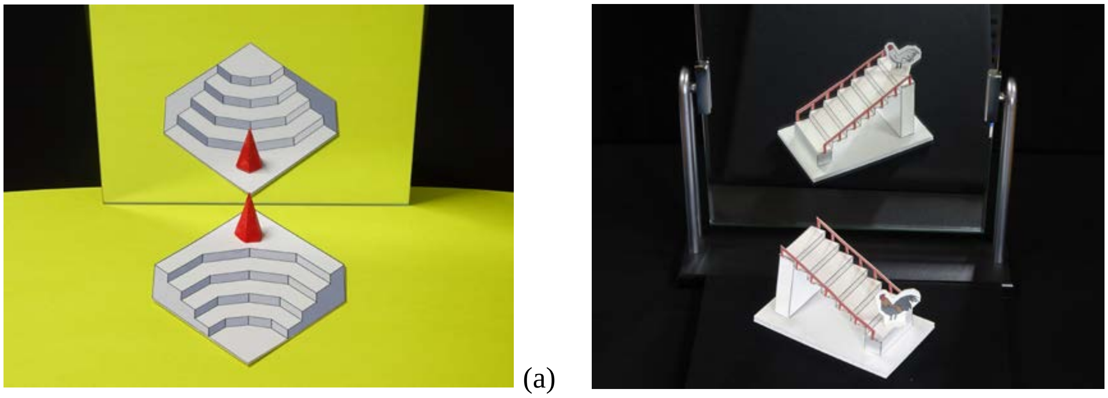
Impossible Objects Created Using Freedom of the Viewpoint

Until now we considered the freedom of depth. As shown in Figure 3, we also have freedom in choosing the viewpoint when we interpret a picture. This freedom creates still other types of impossible objects.

Height-reversing objects are objects whose heights are reversed in a mirror. Figure 11(a) appears as a circular stage in the direct view, but appears as a hill in its mirror reflection. In reality, the object in Figure 11(a) consists of a 2D picture and a 3D red cone placed on it. Interpreting this kind of picture as a 3D structure gives the impression that the object shape changes in the mirror. If the cone is ignored and the image is interpreted instead as a 2D picture, it becomes clear that the direct view and the mirror image are mutually mirror symmetric. Similarly, in Figure 11(b), the staircase seems to change the direction in the mirror. In reality, the staircase is a horizontally placed 2D picture, while the support columns and guardrails are real 3D objects.

These visual effects are based on the next property [9]:

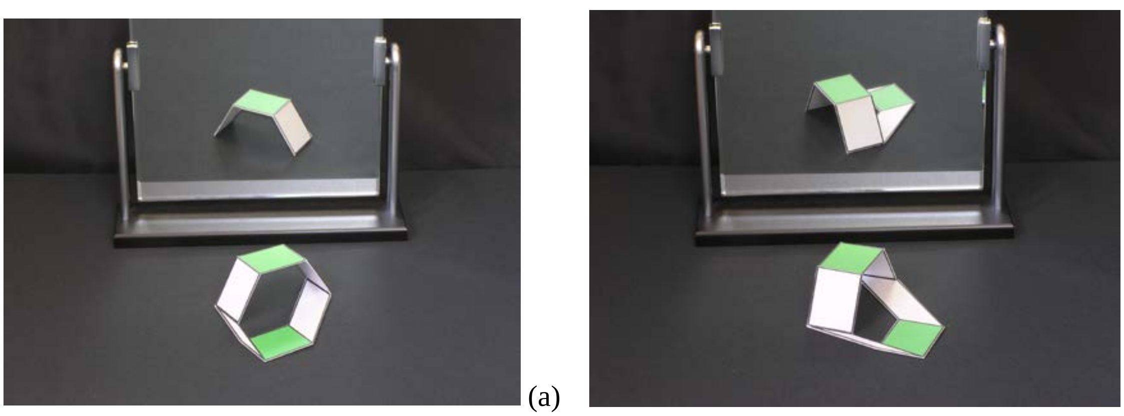
Height reversal property. Let D be a picture of a 3D surface S projected on a horizontal plane at a slanted viewing angle. Then D coincides with the picture of the height-reversed surface of S when it is seen in the viewing direction opposite to the original by the same slanted angle.



(a) (b)

Figure 11: *Height-reversing object.*

Partly invisible objects are objects which become partly invisible in a mirror [10]. Figure 12 shows an example; in (a) a hexagonal cylinder is placed on its side, but the lower half disappears in the mirror. This object consists of a 3D structure for the upper part of the cylinder and a 2D picture representing the lower part, as shown in (b). Because of the height reversal property, the 2D picture coinciding with the upper part in the mirror is consequently hidden completely by the true upper part. This kind of object can be constructed by first choosing a real 3D object that is mirror symmetric with respect to a horizontal plane, drawing its picture on the horizontal plane by projecting it at a slanted angle, and finally replacing the lower or upper half with its picture. When we see the resulting object in the direction opposite to the original direction from the same slanted angle, the 2D picture coincides with the remaining 3D structure and hence one half (upper or lower) of the structure becomes invisible.



(a) (b)

Figure 12: *Partly invisible object.*

Triply ambiguous objects are objects that have three different interpretations when seen from three special viewing angles. An example is shown in Figure 13, where the object is reflected by two mirrors and all the three appearances seem to represent different 3D structures. In reality, the object is a 2D picture with a 3D flag placed on it. The picture represents a rectangular object which has exactly three families of parallel lines. Each of the three viewpoints produces a retinal image in which one of the families of parallel lines is vertical in the retina. Because we see the 2D picture from a slanted angle, the perceived 3D object is compressed along the vertical direction, resulting in the impression that the objects are different. This object won first prize at the Best Illusion of the Year Contest in 2018 [14].

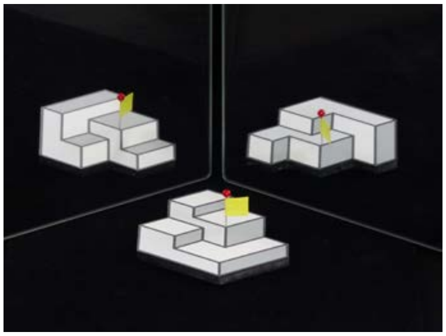


Figure 13: *Triply ambiguous object.*

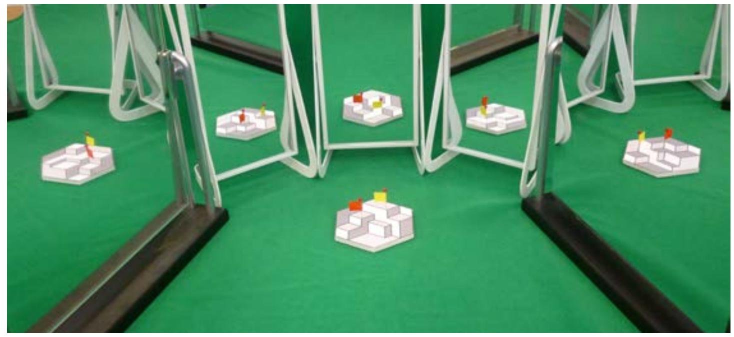


Figure 14: *Ambiguous objects with six interpretations.*

Ambiguous objects with six interpretations are objects that have six different interpretations when seen from six special viewing angles. They can be constructed by combining the principles behind triply ambiguous objects and height-reversing objects. An example is given in Figure 14.

Family Tree of Sugihara's Impossible Objects

We have presented eleven types of impossible objects. On the basis of the geometric properties and optical illusions behind them, here we propose a family tree of impossible objects. The resulting tree is given in Figure 15, where the root is at the upper left corner and the branches extend downward and to the right. The rectangles show the freedom- and perception-related properties, and rounded boxes represent the classes of impossible objects. A tree branch indicates that a child node is created from the parent node, and the symbols by edges represent how the child nodes are created. The symbol "=" means that the same principle is applied to a different way of display. For example, impossible motions are created using the non-rectangularity trick to ordinary pictures. The symbol ">" means that the principle is strengthened by additional constraints. For example, the ambiguous objects are created by applying the picture constraints two times. The symbol "+" means that two principles are combined to create a new class.

This framework gives us new insights. For example, the class of partly invisible objects is an extension of the class of height-reversing objects, although the former was discovered earlier. The lower four classes of objects, the objects made using the freedom in choosing the viewpoints, create visual effects using two steps, firstly by projecting 3D objects onto 2D pictures and secondly by viewing them from different viewpoints, while the other seven classes involve the direct perception of 3D objects.

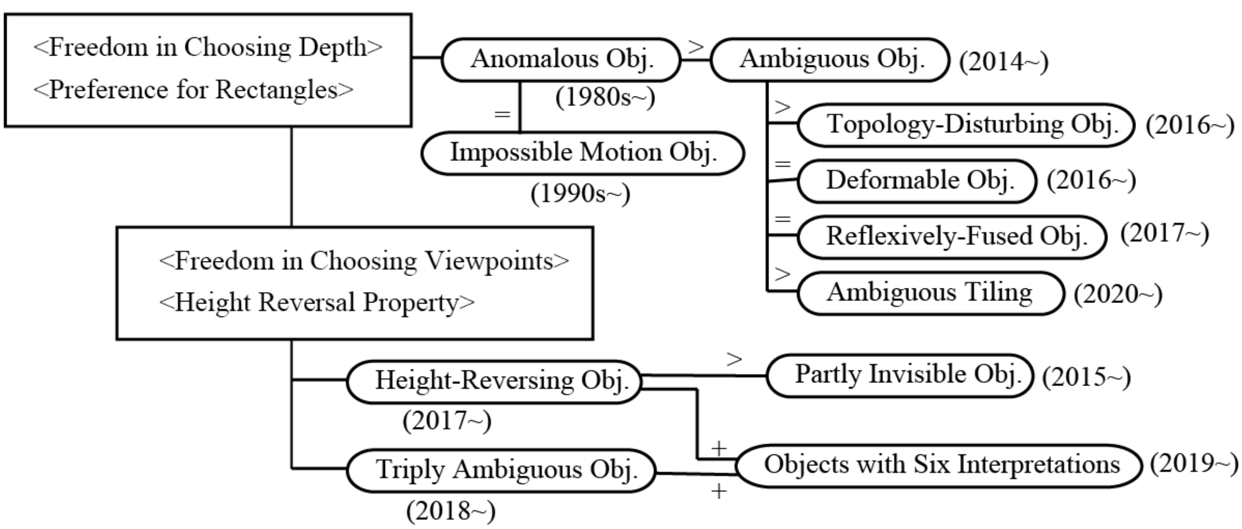


Figure 15: *Family tree of Sugihara's impossible objects.*

Summary and Conclusions

We have reviewed various classes of Sugihara's impossible objects and their mathematical and psychological background, and proposed a new framework to organize them in the structure of a family tree. From this tree, we can visualize the relations among different classes of impossible objects, and can also understand why and how each class of objects creates a sense of impossibility.

This tree may aid us in further studies of impossible objects and 3D optical illusions. Future directions of research include: (1) investigating the mechanisms of creating illusions, (2) developing methods to control the strength of illusions, (3) designing new examples of impossible objects, and (4) finding new additional types of impossible objects. The family tree will guide us effectively to these goals.

For example, one of the possible directions to search for still new classes of impossible objects is to apply the freedom in choosing the viewpoint to 3D objects instead of 2D pictures. A rectangular object, i.e. an object whose faces are planar and connected by right angles, has exactly three families of parallel lines, and hence, the triple interpretations might be generated by placing the object in a way such that one class of parallel lines is projected onto the retina vertically. The height reversal property itself might also be extended to 3D structures, and similarly, the ambiguous objects having six interpretations might be made using 3D objects instead of 2D pictures. Additionally, a new class of "ambiguous impossible motion objects" might be created by combining the principles behind impossible motion objects and height-reversing objects.

Acknowledgments

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