Single-threaded Polyhedra Models

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April 29, 2020

Abstract

We define a class of polyhedra with edge patterns that correspond to two well-known families of knots: torus and Turk's Head. We describe the physical properties of models built with a single loop of stiff wire and classify their 3D shapes. We also present tensegrity models of these polyhedra.

Introduction

Decorative knots, which combine functionality with symmetric and aesthetic qualities, have been used throughout history. To describe these aspects we need to look not only at their topology, the traditional topic of knot theory, but also at their geometry. In this paper we examine the 3D shapes taken by some knots by viewing them as polyhedral models.

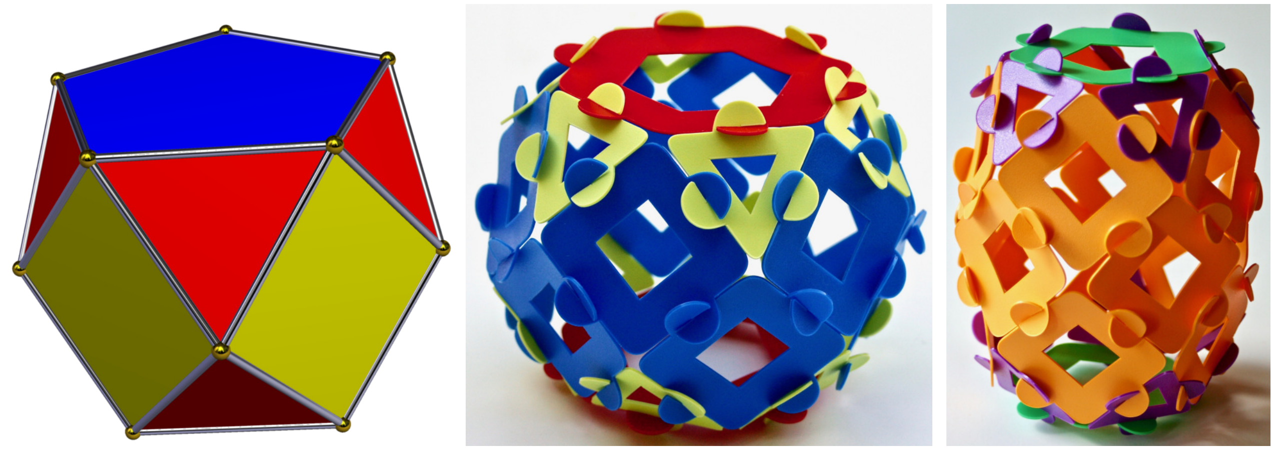
Consider a polyhedron where every vertex has degree 4 (for example, the octahedron, cuboctahedron, any antiprism, etc.) and assume we want to build a wireframe model of this polyhedron. We could accomplish this with a single loop of wire that follows an Eulerian circuit; at each vertex (viewed, intuitively, as a 4-way intersection), the wire may turn left or right or it may go straight through. Assume now that we want to eliminate any sharp bends in the wire by following a path that goes straight through at every vertex. Such a path might close without traversing all edges; if this happens, we can start a new path (and a new loop of wire) at an edge that was not traversed and continue straight through each vertex until the loop closes. Repeating this process until all edges have been visited, we obtain a unique set of circuits called here "straight circuits". A polyhedron model built by tracing the straight circuits with loops of wire is a *link* and each loop is a *knot*. At each vertex we have a choice of crossing the wire either over or under itself; depending on these choices we obtain many links from a given polyhedron. If the wire is interwoven in a perfect over/under manner we obtain an *alternating* link (for example, the three straight loops of the octahedron will be linked as the Borromean rings); polyhedral models built in this manner are described in [2].

We will call the polyhedron "single threaded" if it has only one (Eulerian) straight circuit (i.e., the corresponding link is a knot) and define an infinite class of such polyhedra. We'll classify the 3D shapes taken by the wireframe models which correspond to the *torus* and *Turk's Head* families of knots. We also show how to take advantage of single-threadeness when building tensegrity models of these polyhedra.

There is a growing body of research on the geometric and physical properties of knots, see for example [4]. Stiff wire models of Turk's Head knots are explored in [6] and [7], while general methods of obtaining links from polyhedra are described in [9] and [10]. The opposite of a straight circuit is an *A-trail*, an Eulerian circuit that turns either left or right at each vertex; wireframe models based on A-trails are described in [8].

A Family of Single-threaded Polyhedra

Topologically, the polyhedron $S_{m,n}$, $m, n \geq 3$, has two n-sided polygons as bases, each bordered by n triangles, and separated by $m - 3$ bands of n quadrilaterals each. We will not need here a precise geometric definition,



(a) $S_{4,5}$ (b) $S_{5,6}$ (c) $S_{6,5}$

Figure 1: *Models of some $S_{m,n}$ polyhedra*

but note that we can construct $S_{m,n}$ as a spherical polyhedron that has the symmetry of the right n-gonal prism P_n when m is even and of the right n-gonal antiprism A_n when m is odd (some of the models described below will preserve this symmetry). A few approximate models are shown in Figure 1; in (b) and (c) the ITSPHUN plastic pieces are bent slightly to make the construction possible. Flat-faced polyhedra that have this symmetry exist for $m = 3$ and $m = 4$ (e.g., Figure 1 (a), retrieved from [12]), see below. It should be obvious that every $S_{m,n}$ vertex has degree 4.

We can also define $S_{m,n}$ via *rectification* (critical or complete truncation, *ambo* in Conway's polyhedron notation, see [11]). For any polyhedron P, all the vertices of its rectification aP have degree 4; if P has n edges, aP has n vertices (and thus $2n$ edges), and if P' is the dual of P, $aP' = aP$.

If we set $p = \lfloor m/2 \rfloor - 1$, $S_{m,n}$ is the rectification of either:

- a stack of p copies of P_n (or its dual, a stack of $p - 1$ copies of P_n joined at the ends with two n-gonal pyramids), if m is even; or
- a stack of p copies of P_n joined with an n-gonal pyramid at one end (self-dual), if m is odd.

Thus $S_{3,n}$ is a rectified n-gonal pyramid, which is just A_n, and $S_{4,n}$ is aP_n, a rectified n-gonal prism.

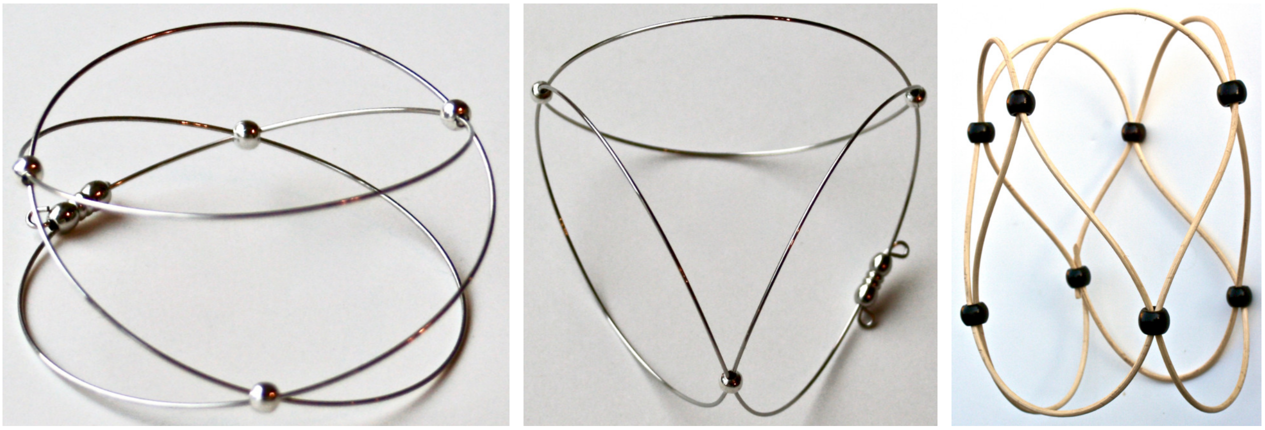
$S_{m,n}$ has $(m-1)n$ vertices and $2(m-1)n$ edges which we can divide into $2n$ base edges and $2(m-2)n$ side edges, which, in turn, can be grouped into $2n$ "diagonals" (pieces of the straight circuit connecting two vertices on opposite bases) of $m - 2$ edges each. There are two families of diagonals depending on the direction they wrap around the side of the polyhedron.

A *straight circuit* in a polyhedron with vertices of degree 4 is a closed path with the property that no two consecutive edges are on the same face. The polyhedron is *single-threaded* if it has only one straight circuit.

Theorem 1. *$S_{m,n}$ has* $\gcd(m,n)$ *straight circuits of equal length.*

Proof sketch. Number the edges of one of the n-gonal bases $0, 1, \ldots, n-1$. A straight circuit that starts with edge i will traverse a diagonal, one edge of the other base, a second, opposite direction, diagonal and return to the starting base with edge $i + m \pmod n$, etc. Each straight circuit will have $n/gcd(m,n)$ such groups of $2(m-1)$ edges consisting of one edge from each base and two diagonals, one in each direction. □

Corollary. *$S_{m,n}$ is single-threaded iff m and n are relatively prime.*



(a) $S_{3,2}$ **(b)** $S_{2,3}$ **(c)** $S_{3,4} = A_4$

Figure 2: *Wireframe models*

We have thus obtained an infinite family of single-threaded polyhedra that includes all antiprisms A_n where n is not a multiple of 3, all rectified prisms aP_n where n is odd, etc. Note that each straight circuit of $S_{m,n}$ completes $m/gcd(m,n)$ revolutions around the central axis of the polyhedron, so if $S_{m,n}$ is single-threaded, its single straight circuit will go around the central axis m times.

Note also that if we include polyhedra with digonal faces, we can extend the definition of $S_{m,n}$ to $m = 2$ and $n = 2$. $S_{m,2}$ has digonal faces as bases, e.g., $S_{3,2}$ (Figure 2(a)), essentially a tetrahedron where two opposite edges are doubled. $S_{2,n}$ is an n-gonal *lucanicohedron*, a degenerate polyhedron with two n-gonal bases separated by n digons (Figure 2(b)). All the results presented here hold for these degenerate polyhedra.

Although there are, of course, many other single-threaded polyhedra, single-threadeness is not common among the most studied polyhedra; for example, none of the Platonic, Archimedean, or Johnson solids with vertices of degree 4 are single-threaded.

Wireframe Models

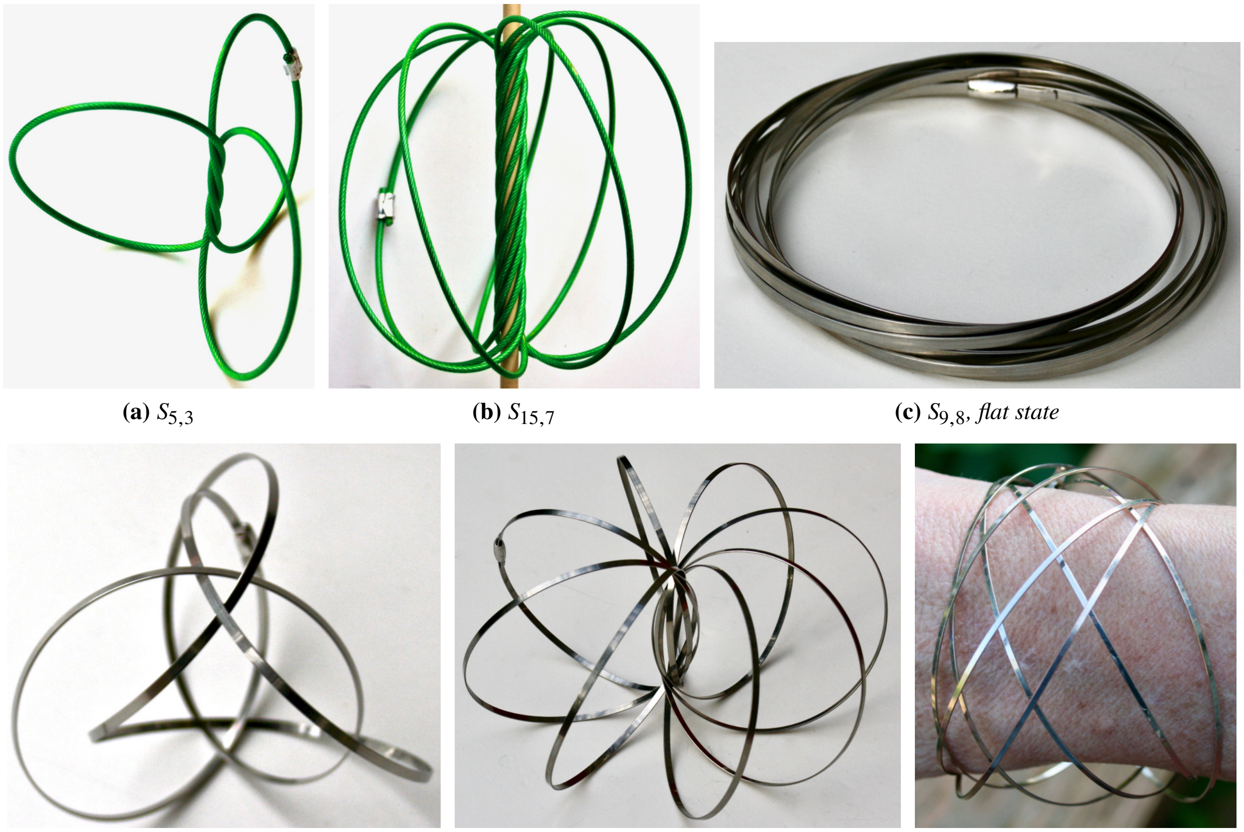
We can build a wireframe model of a single-threaded polyhedron using a single loop of wire that follows the straight circuit. If we build the models in a conventional manner, i.e., tie the wire at the polyhedron vertices, the models will approximate the shape of the original polyhedron, see Figure 2.

We are interested here however in the shape of the models built with a stiff wire that is not tied at the crossing points; this allows the models to adopt 3D shapes that minimize the tension created by the bent and torsioned wire. Such a model can also be viewed as a knot with crossing points corresponding to the polyhedron vertices. Depending on how we overlap the wire at the crossing points, we can obtain many knots, including two well-known families, the *torus* and *Turk's Head* knots.

For the models presented here we used hardened steel wire (music wire and memory wire), steel cable, and round reed, and we noticed that the shapes fall into a few general categories; we summarize these observations in two conjectures.

Torus Knots

Referring to Figure 1, we obtain a model of a torus knot by crossing the wire that follows the diagonals in one direction over the diagonals going in the other direction. (Imagine removing the bases of a model in Figure 1 and wrapping the wire around the resulting torus.)



(a) $S_{5,3}$ **(b)** $S_{15,7}$ **(c)** $S_{9,8}$, *flat state*

(d) $S_{4,3}$ **(e)** $S_{9,8}$, *3D state* **(f)** $S_{7,6}$ *bracelet*

Figure 3: *Torus knot*

The stiff wire models will settle into different shapes depending on how much torsion we force on the wire, i.e., how many times we twist the ends of the wire with respect to each other before joining them together in a loop. Based on empirical observations, we identified two "natural" shapes.

Conjecture 2. *The stable states of a torus model of $S_{m,n}$ with m, n relatively prime and $k = \min(m, n)$, include: (a) a flat coil that wraps around k times, and (b) a 3D shape with k lobes around a central "column".*

A model can be switched between these states with one full twist of the wire ends, see [3] for a demonstration. Figure 3 shows a few models in their flat (c) and 3D (a,b,d,e) state.

The torus knot is symmetric in its parameters, which is hard to visualize since there is no corresponding transformation on polyhedra. This property manifests itself in an interesting way when wrapping a stiff wire around a torus as described above: both $S_{m,n}$ and $S_{n,m}$ models will settle into the same stable shapes, as described in Conjecture 2.

The central column can be forced to open up transforming a 3D model into a flat coil. This coil, which is different from the one in Conjecture 2 (a), wraps around $\max(m, n)$ times and will spring back to the 3D stable state when released. We used this property to make the self-tightening bracelet in Figure 3 (f); a kinetic toy based on this effect, usually called "Flow Ring", is commercially available from many vendors,

In practice, $S_{kn+1,n}$ models can be wrapped in hand (see [3]), while other models require a special jig. The winding of the central column of a 3D model of $S_{m,n}$ tightens as $|m - n|$ increases, which might require making the model larger or using a softer wire (e.g., steel cable in Figure 3 (a, b)).

(a) *Mosaic at Woodchester Roman Villa, AD 325* **(b)** *Cylindrical* **(c)** *Flat*

Figure 4: *Turk's Head knots*

Turk's Head Knots

While torus knots are important in knot theory but not very interesting as real knots, the opposite is true for Turk's Head knots (Figure 4). Clifford W. Ashley devotes an entire chapter of his classic book [1] to this knot and says: "There is no knot with a wider field of usefulness."

Turk's Head is an alternating knot, so the models are obtained by interweaving the wire; (in general, it is always possible to perfectly interweave the straight circuits of any polyhedron with vertices of degree 4). A model of $S_{m,n}$ (in knot jargon, a model of the Turk's Head knot with *m leads* and *n bights*) will settle into one of three possible general shapes: flat (Figures 5(a,b)); (roughly) spherical (Figures 5(c–i)); or tight (rope-like) cylinder (Figure 5(j)). Unlike the torus knot, when making a Turk's Head model we joined the wire ends in a straightforward manner; introducing more torsion in the wire does not lead to any interesting new shapes. The natural state of the flat models is a circular coil that wraps around m times, but, using the friction between coils, they can be arranged into a more interesting semi-stable shape. The 3D shapes have the symmetry of the original polyhedron, modulo the asymmetry introduced by the under/over crossings of the wire. The cylindrical models will elongate and tighten, an effect used in the "Chinese finger trap" toys.

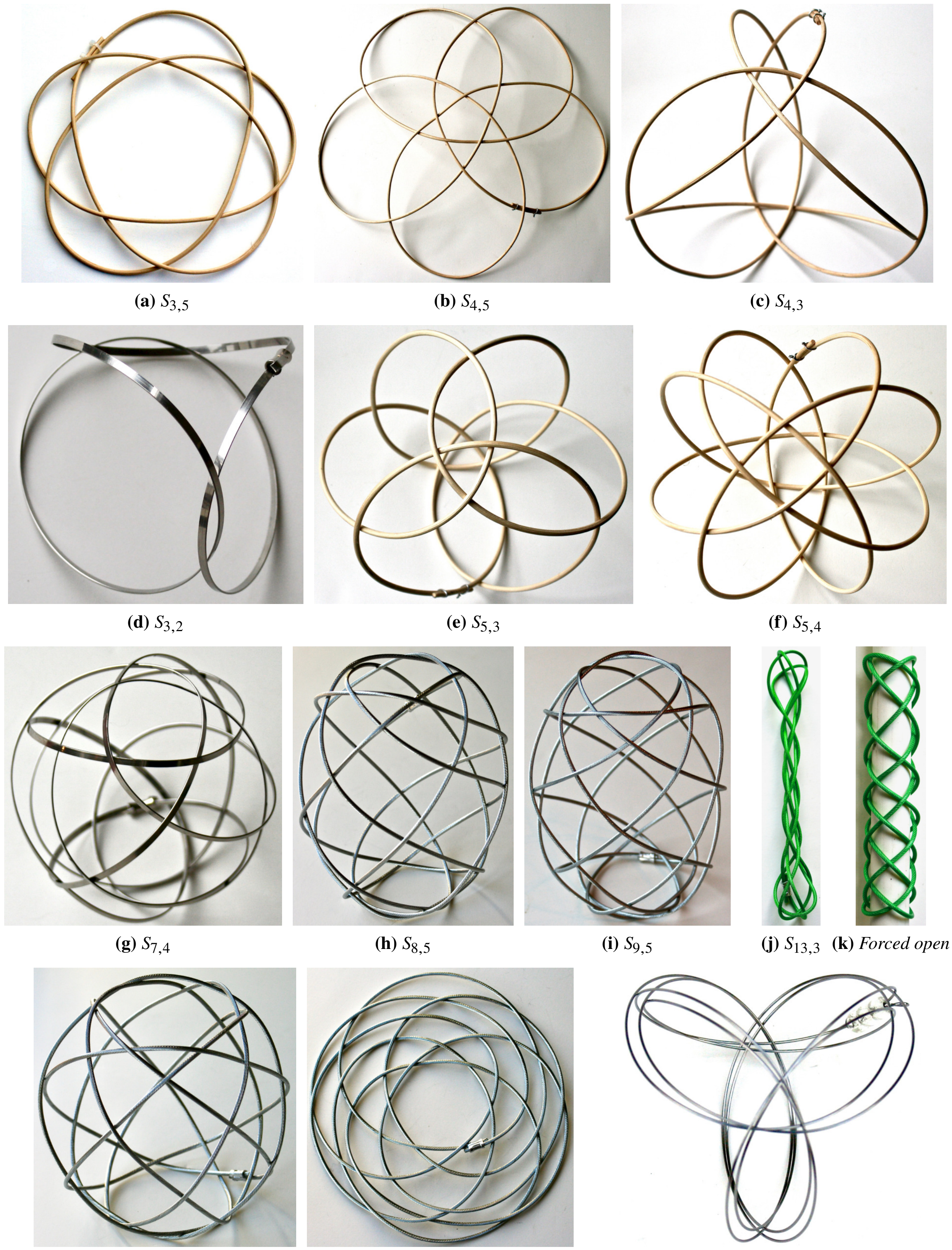
The stable shape is determined by the path taken by the straight circuit after it traverses a base edge and then returns to this base. The model is flat if the circuit returns after less than one complete rotation around the base, spherical if it makes between one and two rotations, and cylindrical if it returns after two or more rotations (for example, follow the circuit from an edge of the central base, around a lobe, and back to the base in the flat, or forcefully flattened, models in Figure 5(a,b,m)).

In other words, since the straight circuit advances m edges every time it returns to the base (see the argument in the proof of Theorem 1), we believe the following is true.

Conjecture 3. *The stable shape of a single-threaded $S_{m,n}$ model (m and n relatively prime) is: flat, if $m < n$; spherical, if $n < m < 2n$; and cylindrical, if $m > 2n$.*

A Turk's Head model is very resilient and will spring back to its stable shape when deformed (flattened, forced to "open" etc.); in Figure 5, for example, (m) will revert to (l) and (k) will revert to (j). Similarly to the torus knot models, this effect can be used to make kinetic jewelry pieces such as rings and bracelets.

By opening half of the model and folding it over itself, the model of $S_{8,3}$ in Figure 5(n) was forced to "double up" from its original cylindrical shape and settle into the stable spherical shape of $S_{4,3}$. In principle, this process can be repeated, turning an $S_{4m,n}$ model into a quadrupled $S_{m,n}$, etc. It might even be possible, to transform an $S_{3m,n}$ into a tripled $S_{m,n}$ by folding into thirds, etc., but we have not tried this.



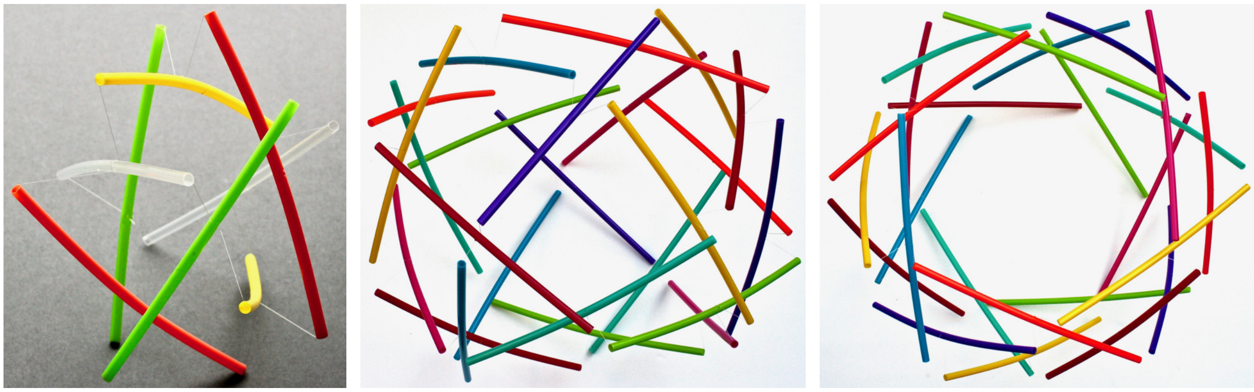
(a) $S_{3,5}$ **(b)** $S_{4,5}$ **(c)** $S_{4,3}$

(d) $S_{3,2}$ **(e)** $S_{5,3}$ **(f)** $S_{5,4}$

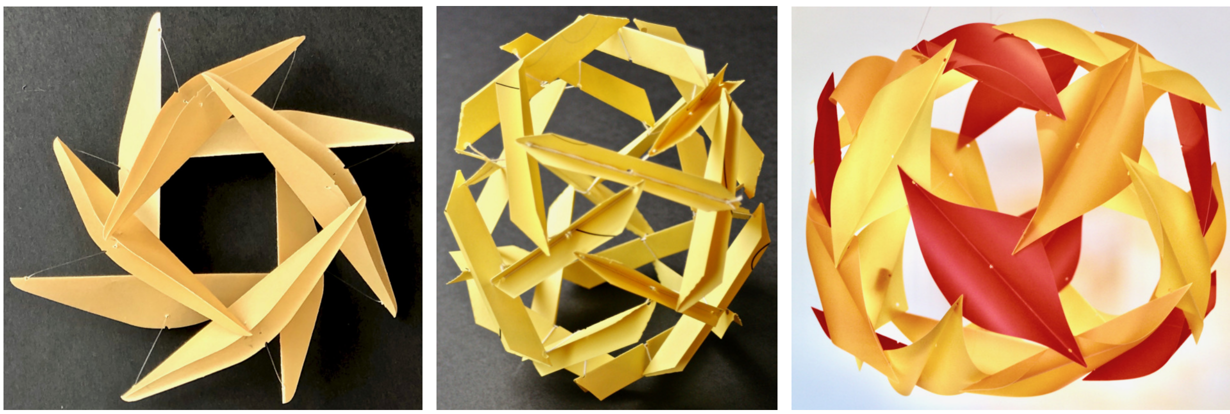
(g) $S_{7,4}$ **(h)** $S_{8,5}$ **(i)** $S_{9,5}$ **(j)** $S_{13,3}$ **(k)** *Forced open*

(l) $S_{7,6}$ **(m)** $S_{7,6}$, *forced flat* **(n)** $S_{8,3}$ *as a doubled* $S_{4,3}$

Figure 5: *Turk's Head models*



(a) $S_{3,4}$ **(b)** $S_{5,6}$ *(compare to Figure 1 (b))* **(c)** $S_{5,6}$ *(top view)*



(d) $S_{3,4}$ **(e)** $S_{6,5}$ *(compare to Figure 1 (c))* **(f)** $S_{5,6}$

Figure 6: *Single-threaded tensegrity models (drinking straws, paper)*

Tensegrity Models

Tensegrity models of single-threaded polyhedra are modular, built by connecting a number of long, (semi)rigid, pieces equal to the number of vertices. Each piece end is connected to a piece middle and each middle is connected to two ends, as described in [2]; the difference is that now, due to the single-threaded property, we can connect all the pieces using a single loop of string that follows the straight circuit of the polyhedron. This string still forms a knot, as described above, but the shape of the model is now given by the arrangement of the component pieces held together by the taut string. Compared to the general method of building a tensegrity model, where the length of the connecting strings (and thus the tension that holds the model together) has to be adjusted separately for each connection, the single string loop allows all the connections to be tightened at the same time, which considerably simplifies the construction process.

A few models are shown in Figure 6. The string connects to each piece twice; it goes along the long axis of the piece (through the straw in the first row pieces) and a second time, perpendicularly, through a small hole in the middle of the piece. The string goes freely through the pieces; by adjusting its length, before tying it into a loop, we obtain the desired tension in the model. This process can be repeated: by loosening the string we can collapse the model (e.g., for transport) and then tighten it again.

Conclusions and Future Work

We identified an infinite class of "single-threaded" polyhedra related to the torus and Turk's Head families of knots. We explored the physical properties and 3D shapes of the models built with stiff wire and presented tensegrity models of these polyhedra.

We plan to continue looking for single-threaded polyhedra with interesting models. A theorem in [5] implies that we will not find any models (knots) with different, non-trivial, symmetries, but, while the variety of symmetries is limited, many more complex braids are possible.

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