Maximizing the Symmetry of Knots

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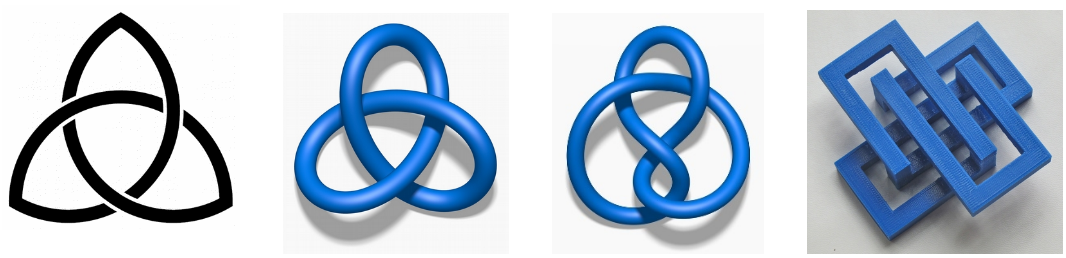
Abstract

Most knots can exhibit more symmetry than is explicitly shown in the Knot Tables. An exploratory approach based on a chord diagram description of a knot permits finding a geometrical representation of a knot with a higher degree of symmetry.

Introduction

The type of geometrical symmetries that prime knots can assume is very limited (e.g. these knots cannot exhibit simple mirror symmetry). Their symmetries are limited to just three families of symmetry groups: "C_n" (Schoenfliess notation [8]) or "*nn*" (Conway's Orbifold notation [7]) exhibits a single *n*-fold rotational symmetry axis; "D_n" or "*nnn*" has the same type of rotation axis, but also has *n* 2-fold rotation axes perpendicular to the primary *n*-fold axis; "S_{2n}" or "*n*X" also has a primary *n*-fold rotation axis, and in addition exhibits glide symmetry, involving a reflection along the primary axis combined with a rotation through 360°/*n* around that axis. Figure 1 illustrates these symmetries with the two simplest knots.



(a) (b) (c) (d)

Figure 1: *Trefoil-knot (knot 3_1): (a) 2D diagram with C_3 symmetry; (b) 3D model with D_3 symmetry. Figure-8 knot (knot 4_1): (c) 3D model with C_2 symmetry; (d) 3D model with S_4 symmetry.*

Despite this, the depiction of all prime knots up to ten crossings in the Rolfsen Knot Tables [5] does not concern itself with symmetry. It turns out that all but one knot (8_{17}) [4] with eight or fewer crossings, and many of the more complicated knots, can be deformed so that they exhibit one of the above geometrical symmetries. Unfortunately, there does not yet seem to be a robust algorithm that automatically finds the maximal symmetry of any given knot. Thus, one has to look for such symmetries by some ad hoc trial-and-error approach. One might try to start with a given projection of a particular knot and apply various Reidemeister moves [6] to shift various trace segments in the knot projection across one another without changing the topology of the knot. But the primary issue is that it is not clear what sequence of moves should be applied to tease out a projection that displays a higher degree of symmetry.

A more promising approach is to form the knot of interest out of a single loop of wire, or of pipe-cleaners, and then manipulate it physically to find more symmetrical configurations; however, there is no guide for how this manipulation should proceed to enhance the chance of finding some hidden symmetry. In the following, we describe a new approach for symmetrizing knots based on the *chord diagram* of a knot, also known as the *Gauss representation* Error: Reference source not found, which is based on the Dowker-Thistlethwaite description (*DT code*) of a knot projection [1, p.35]. In this representation, it is easier to see what moves promise to enhance the symmetry of a given knot.

DT Code and Chord Diagram of a Knot

To obtain the Dowker-Thistlethwaite representation (DT code) of a knot, imagine that the knot is a racetrack and there is a car driving along it, as shown in Figure 2 for the knot 6_3. In this analogy, every crossing in the knot will be a combination of an "underpass" and an "overpass", and the car will pass through every crossing twice before returning to the starting point. The car starts with its "odometer" set at zero, which counts how many crossings it passes through. Every time it drives through an underpass, that underpass is labeled with the value on the odometer. Every time it drives through an overpass, that overpass is labeled with the value of the odometer with a negative sign. Thus, every crossing will be labeled with two numbers. These two numbers will always have opposite parity (one is even, while the other one is odd) [1, p. 35]. After the car has driven once around the whole loop, consider the sequence of crossings with odd numbered labels ±1, ±3, ±5, …, ± (2*n*-1), where the sign is determined by the crossing labels assigned as the car drove once around the loop. At the crossing with label ***k***, we denote the second number as $\boldsymbol{a_k}$ (e.g. look at the first crossing in Figure 2: ***k*** is ***1*** and $\boldsymbol{a_1}$ is ***-8***). The DT code is then defined to be the tuple of integers $(a_{\pm1}, a_{\pm3}, a_{\pm5}, \ldots, a_{\pm(2n-1)})$.

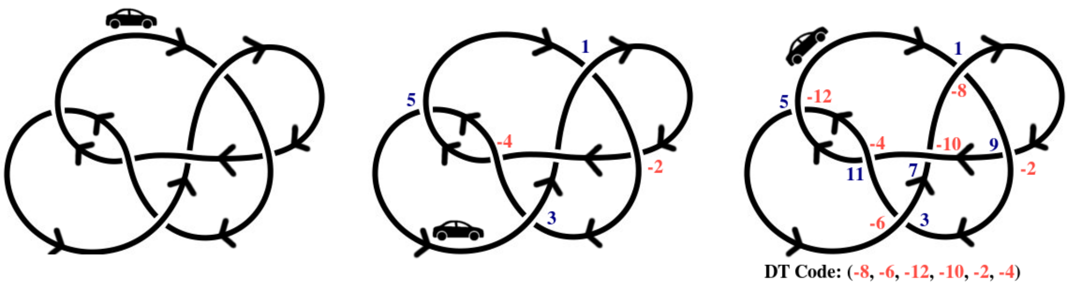
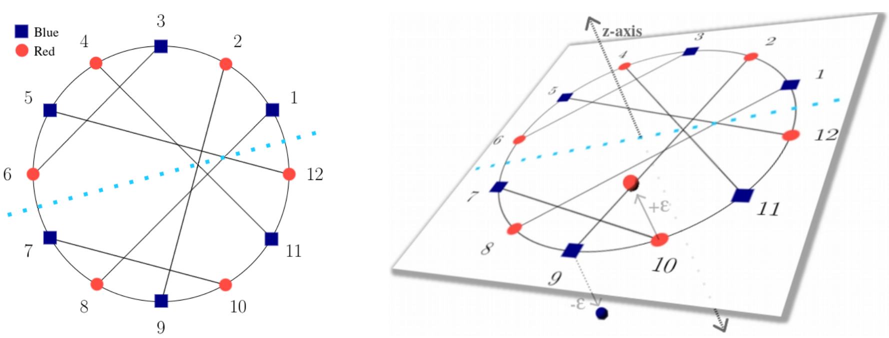


Figure 2: *Constructing the DT code for knot 6_3: (-8, -6, -12, -10, -2, -4).*

This information can now be represented graphically in a *chord diagram*. This diagram is an oriented circle with 2*n* equally spaced *nodes* on it, where *n* is the number of crossings. The circle segments between two adjacent nodes are called *arcs*. In addition, pairs of nodes representing the labels ***k*** and $\boldsymbol{a_k}$, respectively, are connected by *chords* (Fig. 3a). Underpasses are labeled with blue numbers or square dots; overpasses are denoted with red and round dots. A chord always connects a red and a blue node.



(a) (b)

Figure 3: *Chord diagram of knot 6_3: (a) planar view with symmetry axis shown, (b) oblique 2.5D view.*

Our main proposal is that it is easier to enhance the symmetry of a chord diagram than it would be to find the necessary transformation in an ordinary knot projection. However, while it is straightforward to draw the chord diagram for any given 2.5D knot projection, it is more involved to proceed in the other direction [1, p. 36] while also maintaining the symmetry teased out in the chord diagram. Moreover, in order to represent symmetries such as "S_{2n}" in the chord diagram, we need to give it some "thickness." Thus, in Figure 3b, we can imagine that the (dark) blue nodes lie slightly below the *x*-*y*-plane while the (light) red ones lie slightly above. When applying a symmetry transformation to the chord diagram, we also need to make appropriate changes to the colors. If the diagram is flipped through 180° around an axis lying in the *x*-*y*-plane, the underpasses become overpasses and vice versa; all the red nodes become blue, and the blue nodes become red. The same swap is also applied for a mirroring on the *x*-*y*-plane (i.e., negating all *z*-coordinates). For any rotation around the *z*-axis, the node colorings remain unchanged.

Knot Transformations and Symmetry Enhancement

The projection of knot 6_3 shown in Figure 2 exhibits no obvious symmetry, yet its chord diagram has a C_2 flip axis (dotted line in Figure 3a). We could try to manipulate the knot projection into a more symmetrical shape using one of the three *Reidemeister moves* (*RI, RII, and RIII*) [6]. These graphical operations on a knot projection (Fig. 4, top diagrams) do not change the topology of the knot.

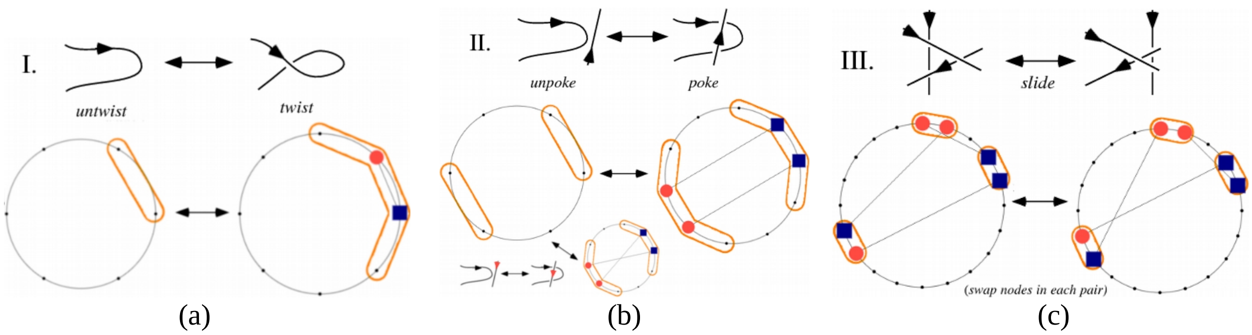


Figure 4: *(a) RI move; (b) RII move (string orientation matters!); (c) RIII move. Top diagrams show Reidemeister moves on strands of a knot; bottom ones show the same moves on chord diagrams. Nodes circled in orange denote the arcs of the chord diagram that are being modified.*

The first Reidemeister move (*RI*) introduces a simple "twist" in a loop (Fig. 4a). *RII* pulls a loop of one strand over or under another strand (Fig. 4b). *RIII* slides a strand through a crossing (Fig. 4c). The lower half of Figure 4 shows how these moves appear on the chord diagram. *RI* simply introduces two new nodes next to one another that are connected by a chord. *RII* introduces a pair or red nodes and a pair of blue nodes, as well as two new chords connecting red and blue nodes. *RIII* takes three pairs of adjacent nodes that form a cycle (through the chords between the tree pairs) and swaps the nodes in each pair.

Looking at an ordinary knot projection, it is clear that these moves can only be applied in a local region, i.e., in a *face* that is not obstructed by other strands (Fig. 5a-c). Intuitively, a face is a connected region in a knot projection that is "surrounded" by *strands*. We will consider the whole region "outside" of the knot (shaded region of Fig. 5c) to be a face as well, which we will denote as the "infinite" face.

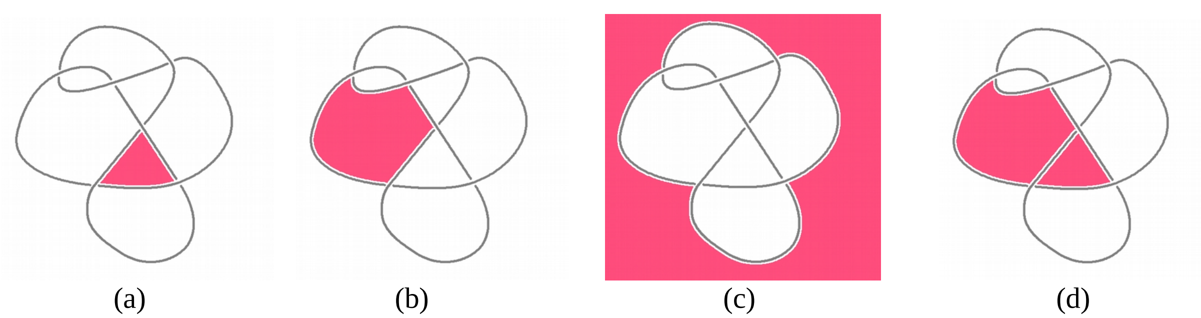


Figure 5: *(a, b, c) Examples of faces in a knot; (d) a non-example of a face.*

In a chord diagram this graphical constraint is more difficult to see. Thus, we need a formal way to define an *unobstructed face*. In the chord diagram, a face is formed by a cycle of edges alternating between chords and arcs without repeating any nodes. The coloring doesn't matter. (There are also some additional criteria that must be satisfied, involving more complicated explanations [1]). An example of a set of node-pairs (i.e. the pair of nodes on an arc) that form a face in Figure 3a are the pairs (1,2), (9,10), and (7,8). To infer a possible symmetry in a knot projection, we need to enhance correspondingly the symmetry criteria in the chord diagram: Not only must the pattern of chords reflect the envisioned symmetry, but in addition, groups of chords and arcs that form faces must also be compatible with the desired symmetry operation.

The criterion for performing any Reidemeister moves in the chord diagram is analogous to the locality constraints in a knot projection. Specifically, an *RI* move must be performed on a single arc. Next, an *RII* move can only be performed in two arcs belonging to the same face. Finally, an *RIII* move can only be performed on a face with three arcs, where one of the arcs has two blue nodes. With the ability to perform "Reidemeister moves on chords," we can search for symmetries solely in the world of chord diagrams, ignoring the final knot projection, until we have reached a state where we have obtained a desired symmetry (or convinced ourselves that the observed symmetry cannot be further enhanced).

Constructing Symmetrical Knot Projections

Here, we discuss the "projection" step, where we construct the knot directly from the chord diagram. First, the outermost contour of the knot projection must reflect the chosen symmetry. There are only three symmetry families to choose from: A C_2 symmetry around a flip-axis in the *x*-*y*-plane; some C_n or D_n symmetry around the *z*-axis; or some S_{2n} symmetry involving a mirroring of the *z*-coordinates. The chord diagram must have one of those symmetries, and it must have a face that maps to itself under that symmetry. This "self-mapping" face forms the start in our method to construct a symmetrical knot projection. We draw an *n*-strand contour with the selected symmetry that demarks the exterior face. We then start adding inner faces that share one or more strands with this outer face. We know that in a valid knot projection, every strand is shared between exactly two faces. This is also a property of a "valid" chord diagram!

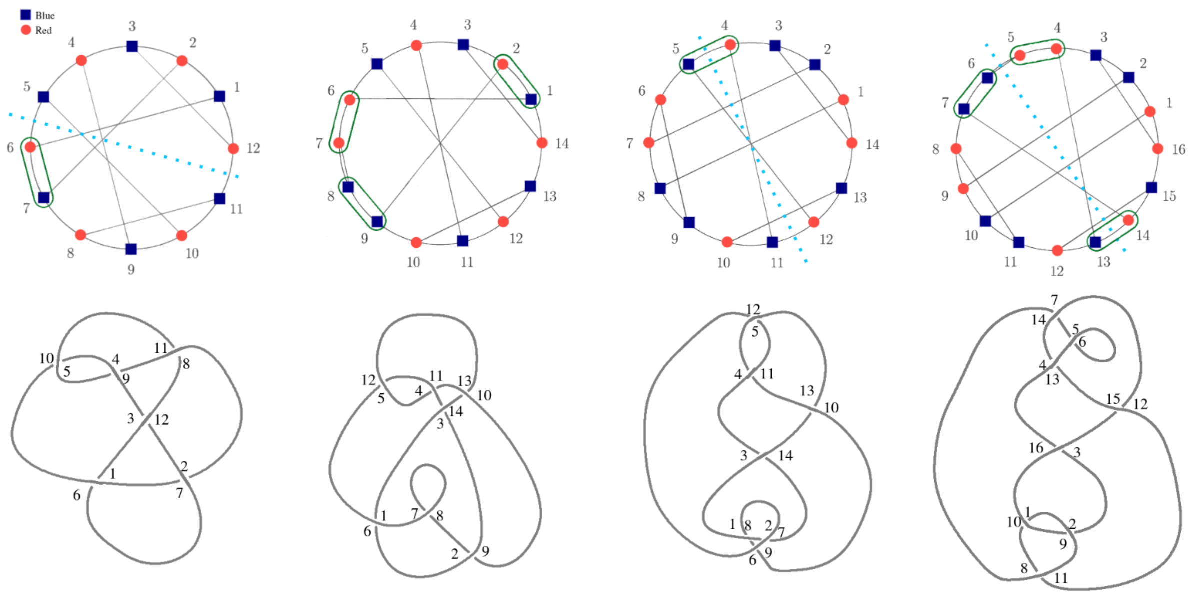
To make the projection step easier, we first find all faces in the chord diagram, label them, and mark the appropriate arcs with their respective two face-labels. After we have drawn the symmetrical exterior face, we mark off this face in the chord diagram. Next, we choose some other face adjacent to the exterior face. By nature of the selected symmetry, this new face either maps to itself, or has other faces that it maps to under the symmetry operation. Again, we mark off the face(s) we just drew in the chord diagram. We continue this process recursively, choosing faces adjacent to faces already marked off, until all faces have been drawn into the knot projection. Consider a simple example: To construct Figure 1a, we would first draw the outside of the trefoil, then add the three petal faces (which map to each other via C_3 symmetry), and finally add the central 3-sided face.

The chord diagram gives the planar layout of the knot projection, but the knot is not defined until all over- and under-crossings have been properly assigned. This information is taken from the red/blue coloring in the chord diagram.

Case Study: Knot 6_3

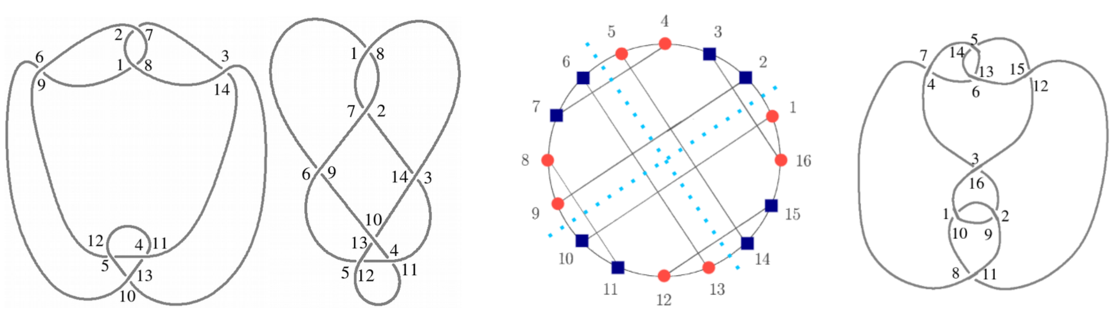
Let's look at this process in detail for knot 6_3. Its depiction in the Knot Table [1, p. 280] shows no symmetry at all, but its chord diagram has a C_2 flip axis (Fig. 6a, top). However, there is no face that maps to itself under this flip-operation. To enhance the overall symmetry of this diagram, we start by introducing an *RI* arc in the chord diagram between nodes 6 and 7. We can then immediately apply an *RIII* chord move to the cycle or circled nodes for the second step (Fig. 6b). Usually, if there is no "obvious" move to make, then adding an *RI* chord can open up opportunities for subsequent *RIII* moves. *RIII* moves can be very powerful, since they allow us to "move" the chords around in the diagram, often revealing new possibilities for symmetry. Additionally, adding *RI* moves helps in shifting groups of nodes to "good" positions in "lopsided" diagrams. For instance, chords {4,9} and {5,10} form an "X"-shape with an off-center intersection (Fig. 6a). This pair can be moved into a more symmetrical position (dotted line in Fig. 6c), if the arc (9,10) is moved to position (11,12) through the introduction of an *RI* between nodes 6 and 7.

This chord diagram does exhibit C_2 symmetry (dotted line in Fig. 6c, top). But, this symmetry is not reflected in the shown projection, because in the bottom half of Figure 6 we simply illustrate the effects on the 2.5D knot projections, as we make the moves in the chord diagram. To realize the symmetry indicated in the chord diagram, we have to follow the procedure outlined in the previous section and choose one of the faces that map onto themselves as the exterior face and draw it in a symmetrical shape compatible with the selected symmetry. In Figure 7 we follow this procedure and show some knot projections that reflect the inherent C_2 symmetries seen in the chord diagram using different faces as the exterior (Figs. 7a, 7b).



(a) (b) (c) (d)

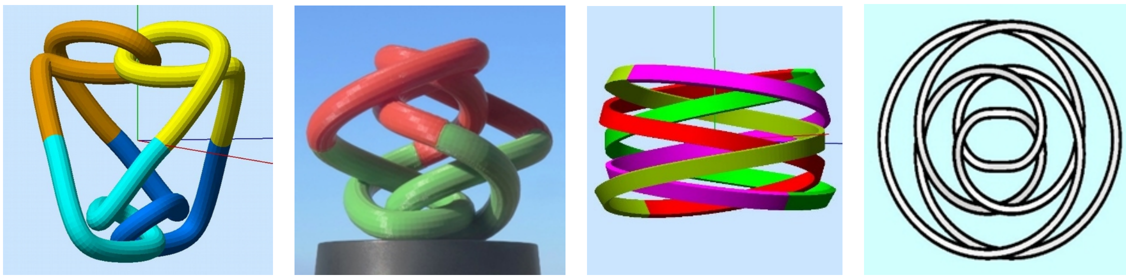
***Figure 6:** First 4 steps in symmetrizing knot 6_3. Top: chord diagrams. Bottom: knot projections. (Darkly circled areas show where the next Reidemeister move will be performed.)*



(a) (b) (c) (d)

Figure 7: *Knot 6_3: (a, b) two alternative C_2-symmetric projections of Figure 6c using different symmetrical faces as the exterior face of the projection. (c) Fully symmetrized chord diagram. (d) A projection corresponding to Figure 7c, implying a "tetrahedral" knot structure.*

Figure 7d shows a projection with a single C_2 flip axis. But, the chord diagram (Fig. 7c) implies that there is more symmetry in this knot; the diagram exhibits four possible C_2 flip axes, and has overall D_4 symmetry. However, there is no true face exhibiting this 4-fold rotational symmetry. This extra symmetry can only be realized in a 3D model. First, the projection in Figure 7d hints at a knot configuration that follows roughly the edges of a tetrahedron, where two opposite edges are formed by pairs of interlinked loops (Fig. 8a). This configuration has overall S_4 symmetry, because one pair of linked loops can be mapped into the other one with a 90° rotation and a mirroring along the *z*-axis. If we prefer, we can model all the strands with smoother, rounded shapes (Fig. 8b); this still maintains S_4 symmetry. This model can then be further deformed into a cylindrical braid (Fig. 8c), which is closely related to Torus-knot(2,5), but with an altered crossing pattern with the sequence (**oouu oouu**)^2 for subsequent **o**ver- and **u**nder-passes. This can be seen more easily, if this braid is flattened into a disk shape (Fig. 8d).



(a) (b) (c) (d)

Figure 8: *Knot 6_3: (a) 3D tetrahedral model, (b) with loopier, rounded strands; (c) corresponding cylindrical braid, (d) flattened into a disk-shaped braid.*

Second Case Study: Knot 8_{13}

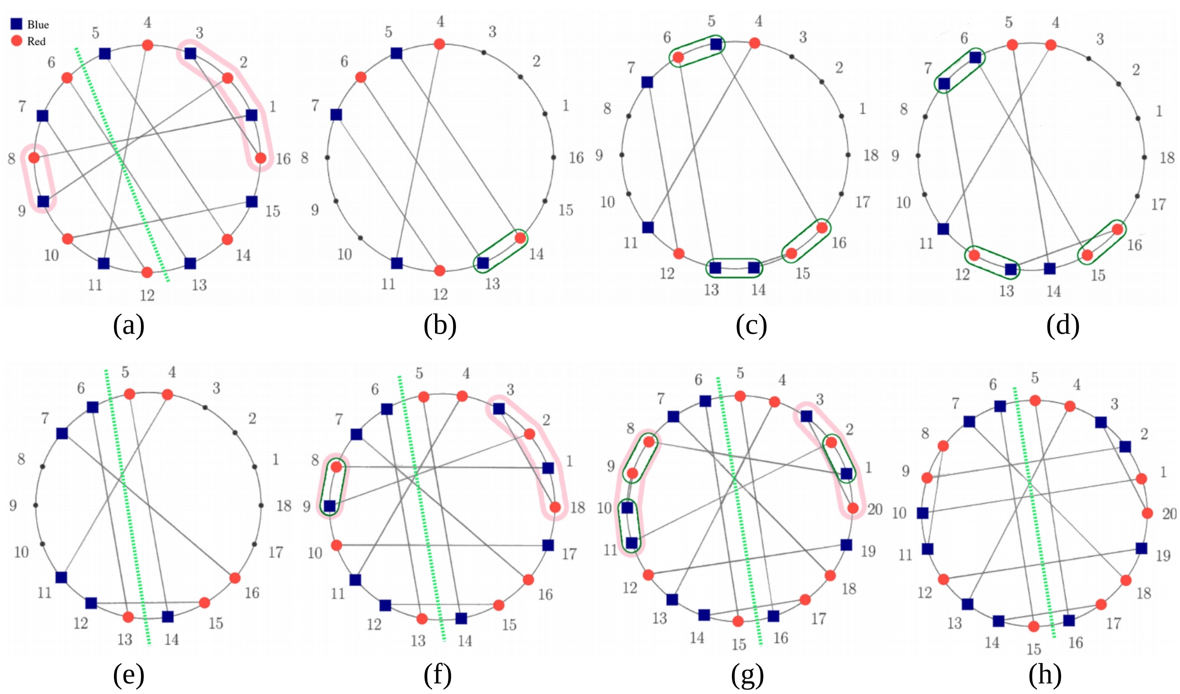
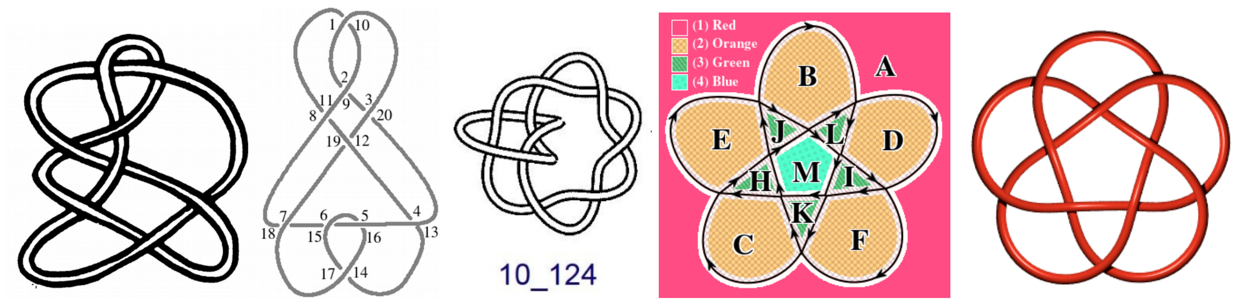


Figure 9: *Symmetrizing knot 8_{13}. Thick lines show target symmetry. Lightly circled areas are easy to fix.*

Knot 8_{13} offers a slightly more complicated example (Fig. 10a). In our discussion of the projection process, we isolate a certain portion of the chord diagram that we want to *try* to symmetrize, while temporarily ignoring all other chords. This is useful when one can see a possible local symmetry within some of the chords of the diagram. In this case, we notice that the chord diagram looks somewhat lopsided w.r.t. nodes 1,2,3,8,9,16 circled lightly in pink in Figure 9a. Similar to what was shown in Figures 6a-c, we can fix this "uneven" balance by using a step also used with knot 6_3: We add an *RI* move on the arc (8,9) so that we can use an *RIII* move later (Figs. 9f-h). Once we know that we can "symmetrize" this portion of the diagram, we look at the larger problem: What chords *could be* aligned with the flip axis shown as a thick green line in Figure 9a. In order to enable this, we ignore all crossings that we know can be symmetrized (or are already in a "good" position). In the steps starting in Figure 9b, we focus on this larger problem area. Note the resulting beneficial effects, as we repeatedly introduce new *RIII* moves. This eventually leads us to the desired symmetric chord diagram (Fig. 9h), which leads to a symmetrical knot projection (Fig. 10b).

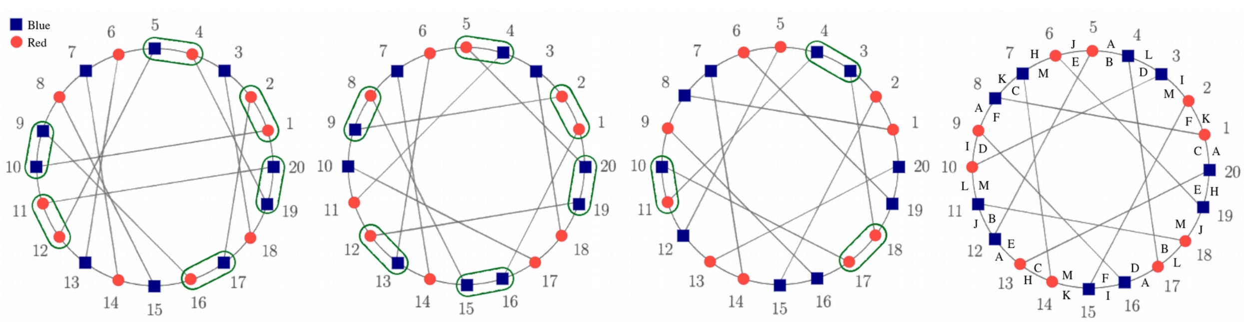


(a) (b) (c) (d) (e)

Figure 10: *Knot 8_{13}: (a) from [1, p.282]; (b) symmetrical version. (c) Depiction of knot 10_{124}: (d) symmetrized version with labeled faces; (d) final symmetrical projection.*

Third Case Study: Knot 10_{124}

A particularly dramatic example of how much symmetry might be hidden in the knot projections in the knot table is knot 10_{124} (Fig. 10c). Figure 11 shows a few snapshots of a sequence of five *RIII* moves that gradually expose the inherent 5-fold rotational symmetry of this knot. In the fully symmetrized chord diagram (Fig. 11d) we can identify a pentagonal face that maps onto itself; node-pairs (1,20), (8,9), (16,17), (4,5), and (12,13) form such a face. To obtain a corresponding knot projection, we make this pentagon the external face of the knot. We then attach to its inside five identical faces (Fig. 10d), and we continue this process of inward-fitting additional faces, adding faces in the order: red, orange, green, blue in Figure 10d, while maintaining strict 5-fold rotational symmetry. After all twelve faces have been placed, a highly regular structure emerges (Fig. 10e). By analyzing the over- and under-passes, we realize that this is the Torus-knot(5,3), projected with 10 crossings. Considering its 3-dimensional nature, we note that this model has D_5 overall symmetry, with five C_2 flip axes in the *x*-*y*-plane.

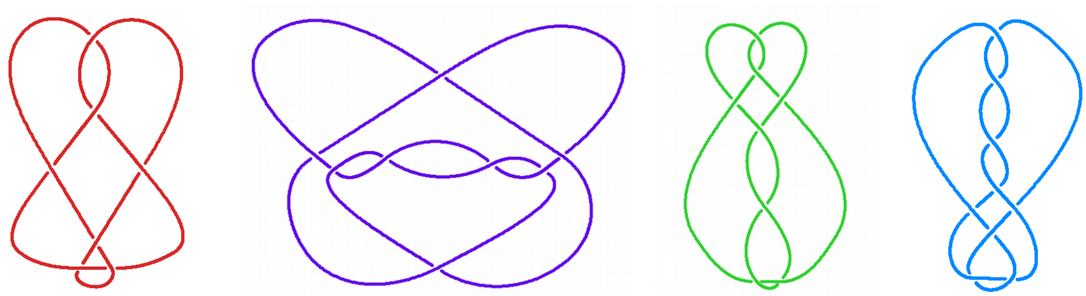


(a) (b) (c) (d)

Figure 11: *Symmetrizing knot 10_{124}: (a, b, c) multiple RIII moves; (d) symmetrical chord diagram with its arcs labeled with the faces they belong to.*

Results and Discussion

Using chord diagrams for the purpose of knot symmetrization serves as a promising avenue for finding highly symmetrical, aesthetically pleasing representations of knots. We found that many more knots have inherent symmetries than one would conclude from looking at the knot tables. Specifically, we were able to find symmetries (Fig. 12) for all but one knot with eight or fewer crossings (knot 8_{17}). Naturally, we wondered why knot 8_{17} did not yield any symmetry at all. We believe that this is related to the fact that this is the first knot that is negative amphichiral [4]. All other knots up to this point are reversible and/or amphichiral.



(a) (b) (c) (d)

Figure 12: *A selection of knots that have been symmetrized with our approach: (a) knot 7_6; (b) knot 8_4; (c) knot 8_{12}; (d) knot 8_{13}.*

There were other surprises. We previously thought that knot 6_3 was the first knot that did not have any symmetry at all. But, by working with the chord diagram, we were able to find several projections with seven crossings that exhibit C_2 flip symmetry (Figs. 7a, 7b). By pushing the chord diagram processing even further, we also found an 8-crossing projection that not only exhibits C_2 flip symmetry (Fig. 7d), but which also suggests a 3D-presentation of this knot that exhibits S_4 symmetry (Figs. 8a, 8b). When projected onto a sphere, both these models would also show eight crossings. The model shown in Figure 8b is basically a cylindrical braid (Fig. 8c); it can be flattened into a planar disk-like braid, which also exhibits eight crossings.

Another advantage of using the chord diagram is that it allows for a concise symbolic description of the Reidemeister moves. In principle, this allows us to describe and execute these transformations in a computer program. However, we do not yet have a clear procedure that tells us which are the best Reidemeister moves that lead to maximal symmetry; thus our searches were somewhat ad hoc and possibly incomplete. Letting a computer program do this search, would allow for an exhaustive brute-force search that might reveal additional symmetrical knot presentations.

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