Design of a Sampler of Isohedral Tilings of the Pied-de-poule Tile

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Abstract

We describe the design of a sampler of all isohedral tilings of the classic pied-de-poule tile. The sampler is a Hawaiian shirt, also known as Aloha shirt. This allows us to explore the wider applicability of the pied-de-poule motif, yet respecting the design principles of classic Hawaiian shirts. We formulate the precise inclusion and exclusion criteria of the set of tilings to be enumerated. By checking the various combinations of symmetries, we find that there are eight distinct tilings, to be included in the sampler. We also show representations of the three orbifolds of the corresponding symmetry groups, which are added to the shirt in the form of phantasy sea creatures, based on a torus, a Klein bottle and a twisted pillow.

Introduction

A sampler is a one-piece artwork in which all instances of a well-defined and finite class of patterns are represented [6]. It is like a gallery of patterns, packaged as one piece. Our goal is to design a sampler showing all isohedral tilings (tessellations) constructed by the repetition of the basic pied-de-poule tile. This basic tile is defined in [3]. Pied-de-poule, also known as houndstooth, is a classic weaving pattern, frequently used in fashion. The basic tile can be tiled in infinitely many distinct patterns, but here we restrict ourselves to isohedral tilings: each copy of the basic pattern is embedded in the same way in the pattern. How many such tilings can be made? We shall demonstrate that there are eight different isohedral tilings. The symmetries of a tiling belong to one of the 17 plane crystallographic groups, also known as wallpaper groups. The eight isohedral tilings belong to three of the plane crystallographic groups. Nowadays these groups are denoted in the orbifold notation developed by Conway [2]. In our sampler, we like to show representations of the orbifolds, not just their notation (∘, ×× and 2222).

Our sampler will be inspired by the Hawaiian shirt, also known as Aloha shirt. Thus we explore the wider applicability of the pied-de-poule motif.

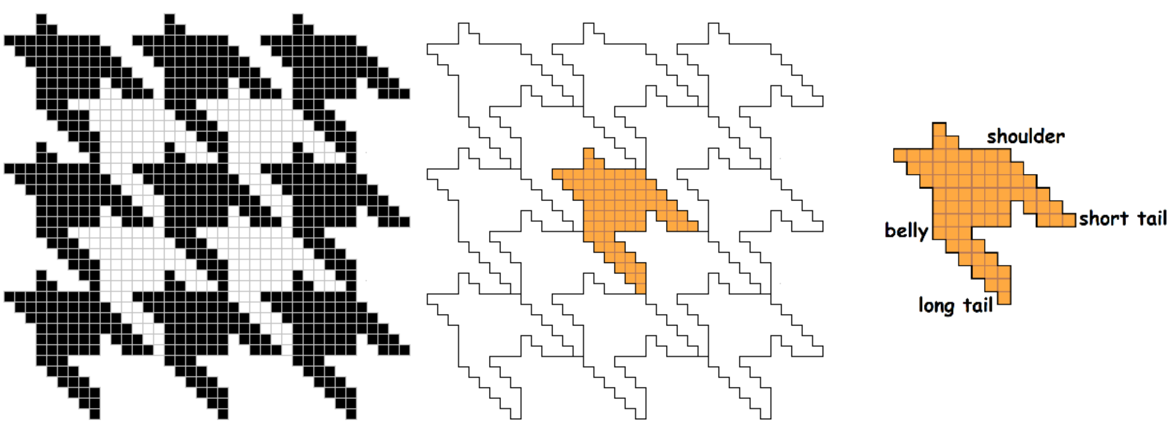

Figure 1: *Classic pied-de-poule pattern of 13 tiles (left). Essential tiling (center). One basic tile (right).*

We show the classic pied-de-poule tiling in Figure 1. Each tile touches only two other same-color tiles and the touch length is the side of one grid cell. In Figure 1 (center), we remove the coloring to show the essential tiling, highlighting one copy of the basic tile (amidst 12 other copies). We introduce some names for the body parts of the basic tile (Figure 1, right). There is not just one pied-de-poule [3], but an indexed family, one pattern for each N > 0. Here we use N = 3, the same principles work for any N > 1.

If the tile is taken to be a puzzle piece, the question arises what kinds of patterns can be created by arranging the pieces. We restrict ourselves to *tilings*: no overlap and no gaps. We further restrict ourselves to *monohedral* tilings (only one basic tile). It is allowed to use mirrored, translated, rotated and glide-reflected versions of the basic tile. The pied-de-poule tiles interlock [3], because the tips are caught between the tails of the next tile. The tiles necessarily form strips, which can have an arbitrary length. We aim at infinite tessellations, working in the Euclidean plane, which extends infinitely far in both directions. Thus these strips appear in every tessellation and they all have a parallel orientation. The direction of the translation vectors relating the tiles inside a strip can be chosen arbitrarily. Without loss of generality, we choose to orient the vector along the diagonal $x + y = 0$ in the usual Cartesian coordinate system. If the tips point upward and left, we call it a strip in A orientation, if the tips point downward and right, we call it B. If one tile in the strip is mirrored, all tiles in that strip are mirrored, as the interlocking enforces this. We denote such strips as A_m (tips upward and left) and B_m (otherwise).

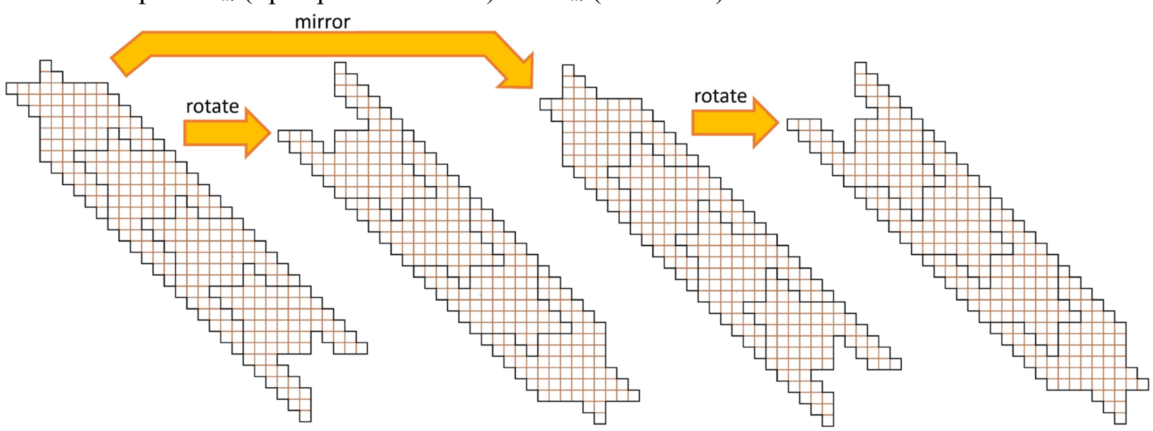

Figure 2: *Strips with orientation A and B. Mirrored strips with orientation A_m and B_m.*

The five strips in Figure 1 (center) have the A orientation, for example. In Figure 2, all four orientations are shown. For the relative positioning of two adjacent strips, there are always six essentially different possibilities, sliding along the staircase formed by the strip's boundary. An uncountable number of tilings is possible. This is far too much for our sampler and we adopt further restrictions: the tilings should be highly symmetric and another restriction is that they are as pied-de-poule-like (houndstooth-like) as possible. Before we can formulate such restrictions in a precise manner, we need a few definitions.

Enumerating the Isohedral Tilings

We aim at classifying all tilings, where we adopt the constraint introduced by Heesch and Kienzle [7] under the term *regelmäßige Zerlegung*. It is called an isohedral tessellation. A tiling is said to be *isohedral* if any pair of tiles are equivalent under a symmetry of the tiling. In other words: each tile meets its neighbors in precisely the same way. A non-isohedral tiling is, in an intuitive sense, less regular than an isohedral, because distinct copies of the tile play different roles. Kaplan [9] writes: "The isohedral tilings play a valuable role in art and ornamental design. They correspond to an intuitive notion of regularity in monohedral tilings: every tile plays an equivalent role relative to the whole."

Pied-de-poule-like means that it is possible to color the tiles in two colors (black and white, without loss of generality). A *coloring* of a tiling is a mapping from all its tiles to {black, white}. We insist that adjacent tiles in each strip are black and white in an alternating manner. This rule prevents the appearance of elongated white tiles, or elongated black tiles, which are not pied-de-poule-like at all. Next, we avoid that adjacent tiles have a long same-color touch range. In the classical pied-de-poule, each tile has two contact points of size one (one at its long tail tip, one at its shoulder). For a given tiling and a given coloring, we call the sum of the length of the edges a tile has in common with same-colored neighbors its *contact* *size* (the unit is the side of a grid cell). It is easy to increase the contact size by shifting the tile's neighboring strip further down the staircase edge, but those solutions are *not* pied-de-poule like and have to be rejected. We call a tiling *minimal-contact colorable* if it allows for a coloring in which the interlocked tiles are alternatingly colored and each tile's contact size is at most two units (see Figure 3).

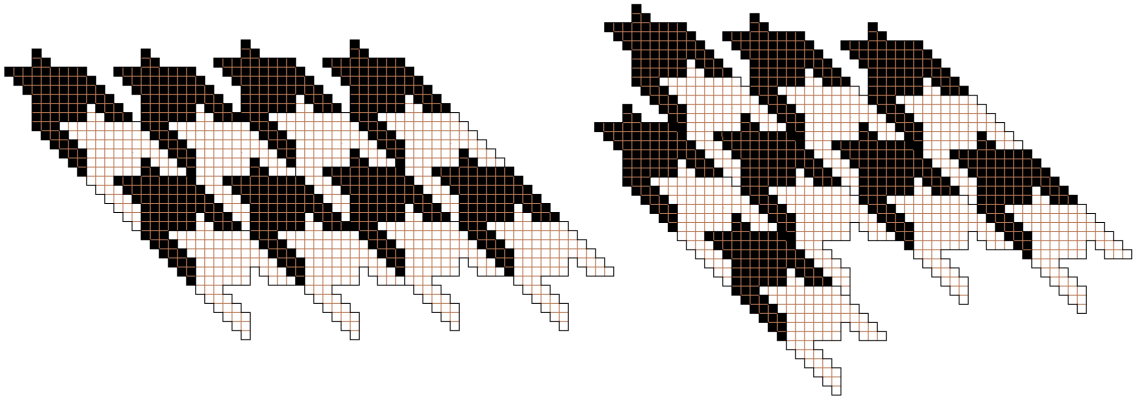

Figure 3: *Tile configurations to be rejected. Left: not isohedral, right: not minimal-contact colorable.*

Now we want to create an overview of *all* isohedral minimal-contact colorable tilings of our basic tile. We only need to experiment with three adjacent strips: once these are positioned, the positions of all remaining tiles are determined by the symmetries in the configuration of these first three strips. Without loss of generality, we freely choose the orientation of the first strip to be *A*. For the second strip we have four possibilities: the same (*A*), rotated over 180º (*B*), mirrored (A_m) and mirrored and rotated (B_m). The third strip has the same direction as the first strip again[1].

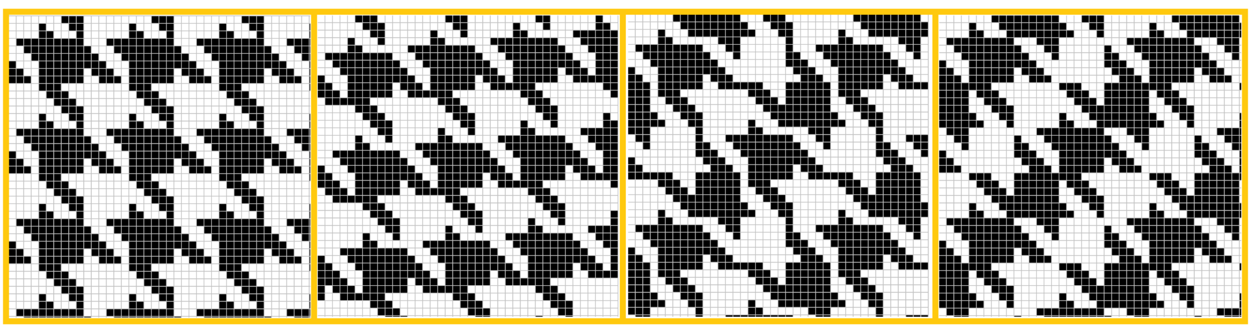

Figure 4: *Isohedral minimal-contact colorable tilings. From left to right: classic pied-de-poule, modified pied-de-poule (both type TTTTTT), tail-to-tail pattern, shoulder-to-belly pattern (both type $TG_1G_2TG_2G_1$).*

At the staircase contact line between two strips, there are six different positions to place the next strip (12 after coloring), but we find that at most two of them are minimal-contact colorable. Contact size of less than 2 does not occur. If there are neither rotations nor glide reflections (strip orientations *AAA*) we find

[1] For consider otherwise, i.e., suppose there is an isohedral tessellation where the 3^d strip is *not* a translated version of the 1^st. Then the 3^d strip is either a rotated, a mirrored or a rotated & mirrored copy of the 1^st. All 3 cases leads to a contradiction. First case (rotated): take an arbitrary tile in the 1^st strip and one in the 3^d. There is a rigid movement from the former tile to the latter: a rotation. The rotation center is *between* the 1^st and 3^d strips. As the tiling is isohedral, this rotation must be a symmetry of the tessellation. Thus there must be a rotational symmetry whose center is somewhere in the middle strip. This symmetry leaves the entire plane invariant, also the middle strip. But a strip has no rotational symmetry (the tile has no rotational symmetry and the interlock alignment is translation-based). This is a contradiction. 2^nd case (mirrored): the 1^st and 3^d strips are related by a glide reflection, reflection line parallel to the strips. The reflection line lies in the middle strip, which however has no (glide) reflection. Again, a contradiction. 3^d case (rotated & mirrored): the two outer strips are related by glide reflection, perpendicular reflection line. The middle strip should be invariant under the glide reflection; but it is not: again, a contradiction.

two good solutions, one of which is the classic pied-de-poule tiling [3]. Both have Heesch type TTTTTT (see [7] for the Heesch types, T means translation C twofold rotation, G glide reflection). If there are rotations but no glide reflections (strip orientations *ABA*) we find four solutions (all four having Heesch type TCCTCC). One of these was reported in Bridges 2012 [3], three are new. If we have no rotations but glide reflections (strip orientations AA_mA) we find no acceptable solutions. For the case of rotations *and* glide reflections (strip orientations AB_mA) we find two solutions (both $TG_1G_2TG_2G_1$). One solution was in [4] and [5], the other is new. Thus our enumeration yields 8 tilings (Figures 4 and 5).

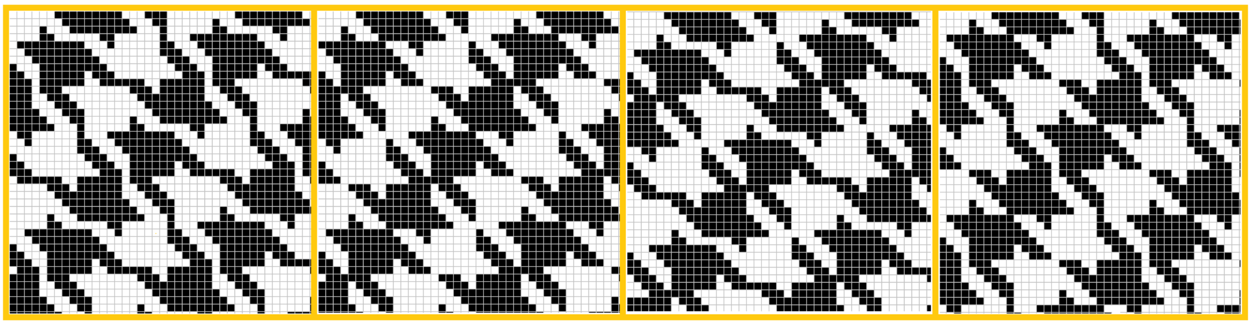

Figure 5: *TCCTCC tilings. From left to right: tail-to-tail pattern, body-to-body pattern, mixed pattern (belly-to-belly/short-tail-to-short tail), second mixed pattern (shoulder-to-shoulder/long-tail-to-long-tail)*

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Constructing the Orbifolds

Besides the eight patterns themselves, we also like to highlight the symmetry groups. First, we note that our eight tilings belong to three tiling types [7]: Heesch types TTTTTT, TCCTCC and $TG_1G_2TG_2G_1$. Tilings are special cases of wallpaper patterns and the symmetries of each wallpaper pattern form a group, which is a plane crystallographic group. In general, there are 17 such groups; in our case, we have three groups, viz. p1, pg, and p2 in IUCr notation, or ○, xx and 2222 in orbifold notation.

There is a modern connection between tiling theory and topology. It describes the essence of a symmetry group as a 3D object. Thurston proposed a concept called orbifold, promoted by Conway, who also developed the orbifold notation [2]. The *orbifold* of a plane crystallographic group is formally defined as the quotient of the Euclidean plane under the group actions. In practice, this means that we cut or fold the unit cells of the tiling, put them on top of each other and identify corresponding elements.

In Figure 6 we identify corresponding elements, here for the first pattern of Figure 5. The centers of rotation are indicated by small circles and the arrows indicate for each protruding corner of the tile where its corresponding corner would be on the next tile (but here put back onto the same tile).

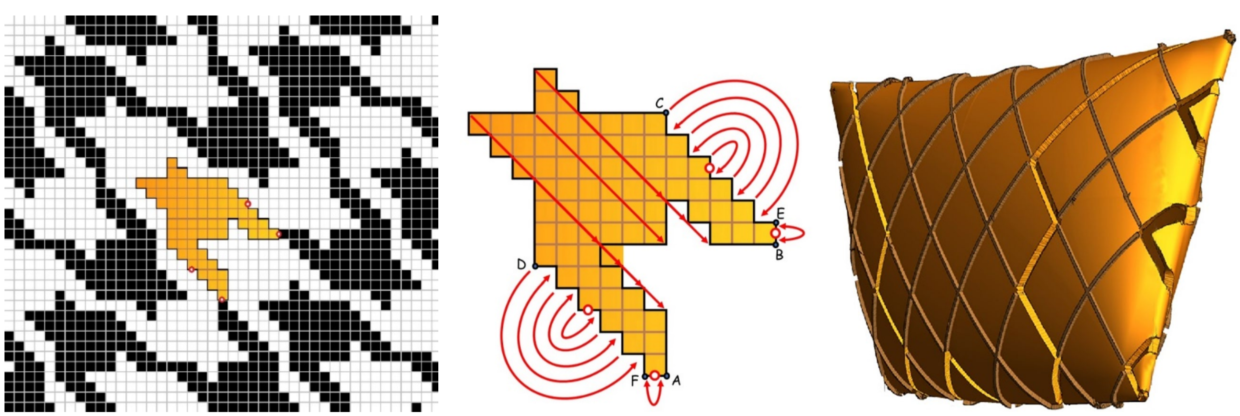

Figure 6: *A tile in relation to its neighbours in TCCTCC tiling (left). Identification of corresponding contour elements (center). Mathematica rendering of the corresponding one-tile orbifold (right).*

The theory of orbifolds tells us to *identify corresponding edges*, which sometimes is hard to do in physical reality. In topology, we can say: if we identify the opposite sides of a rectangle we get a torus. But the orbifold itself is a concept based on metric geometry (rotations, translations, etc. preserving distance). Saying that an orbifold is a torus formally means that it is homeomorph to a torus (it can be morphed by rubber transformations). Another way to understand orbifolds is developed by Hilden et al [8], who view them as artifacts for stamping symmetric designs. Yet another perspective was developed, at least for two specific 3D forms, in [4] where a torus and a Klein bottle were presented as a (pied-de-poule) tiling with just a single copy of the tile. The latter work preserves the staircase nature of the tile's contour, working with diffeomorphisms rather than homeomorphisms (stretching, yet avoiding folds, wrinkles, and sharp edges). Here we adopt the latter perspective. Figure 7 shows three 3D printed objects, one for each of the three types of tiling patterns we found in the previous section.

These 3D objects are described by equations, rendered in Mathematica. If we interpret the tile's boundary, shown as a cut-line, as edges-to-be-identified, then the objects of Figure 7 approximate the orbifolds of our patterns, or more formally: they are diffeomorphic to the orbifolds of our patterns (except at the twisted pillow's cone points). A purely topological perspective is not precise enough (using the homeomorphisms of topology, a twisted pillow and a sphere are indistinguishable). A *diffeomorphism* is a continuous and differentiable mapping, whose inverse exists and is also continuous and differentiable.
For the torus and the Klein bottle, we refer to Bridges 2018 [4]. The twisted pillow is new. It corresponds to the tail-to-tail TCCTCC pattern of Figure 6 (left). The other TCCTCC patterns have a slightly different pillow. The pillow's diffeomorphism does not introduce much distortion: it is *almost* possible to iron the surface out again and get a flat tile without much stretching. This is different from the other two objects, the Klein bottle and the torus, where the grid and contour are seriously deformed.

The twisted pillow orbifold posed novel challenges because of the singularities at the four corners of the pillow. Each corner locally is a cone with a top angle of 90º [2]. Experimenting with paper models of edge-glued pied-de-poule basic tiles we found that the overall twist of the pillow is about 20º when inflated, 45º when flat. A pillow can be inflated to a thickness of at least 50% of its width (25% of length). Some distortion seems unavoidable: the inflated paper models buckle a bit and then have both positive and negative curvature, for example, a saddle point appears in the middle of each of the four edges. A developable surface is also possible, in which case we have two sharp folds. The twisted pillow of Figures 6 (right) and 7 completes the journey began when we designed the torus and the Klein bottle.

For our sampler, we give the orbifolds the form of phantasy sea creatures (Figure 8).

Figure 7: *Klein Bottle, Torus, and Twisted Pillow models which are diffeomorphic to the basic tile. When closing the cut gap they are diffeomorphic to the orbifold of the corresponding tiling.*

Design Principles and Implementation of the Sampler

Hawaiian shirts have been developed since the early 1900s and are the result of commercial and touristic interaction between Hawaii and the USA (with a bit of Japanese influence). The Hawaiian shirt has been popularized by celebrities such as Elvis Presley and Tom Selleck. At the same time, it is a stereotype image for leisure time and holidays. We refer to [10] for an overview, [12] for examples and [11] for a semiotic study (*semiotics* is about signs and meaning). The shirt has status in Hawaii, in the communities of surfers, of rockabilly music and in the office on "casual Friday", which began as "aloha Friday". At the same time, the shirt can also be a symbol for bad taste, for example in cartoons featuring overweight tourists. Marcia Morgado describes this type of ambiguity [10]: "the shirt derives its contemporary meaning from the play between binary oppositions: the transient played against the durable, the copy played against the genuine, and the tawdry kitsch aesthetic played against the indigenous folk art aesthetic." Aloha shirt is taken to be a synonym of the Hawaiian shirt. Usually, they are printed in cheerful colors, often showy. Typical themes are palm trees, hibiscus flowers, parrots, palm trees, pineapples, surfboards, the ocean, fish, turtles, boats, hula dancers, ukulele, "tiki" statues, cocktail glasses, etc. The sleeves are short; usually, the shirt has a flat collar and coconut buttons.

We work towards a type of Hawaiian shirts sometimes called "Tapa", although the original term refers to Polynesian bark-cloth fabrics, before the Hawaiian shirt. Tapa shirts have lots of geometric motifs in brown and ochre colors and island themes such as sea creatures (fish, turtles), shells, tropical plants. The geometries and the plants/animals are organized in an overlapping system of rectangles.

Our design is meant to stay close to this Tapa tradition, with square regions of pied-de-poule variations, fantasy creatures, and two more figurative graphics. The fantasy creatures (Figure 8) are a playful interpretation of the orbifolds, playing the role of the fish and turtles of traditional Hawaiian shirts.

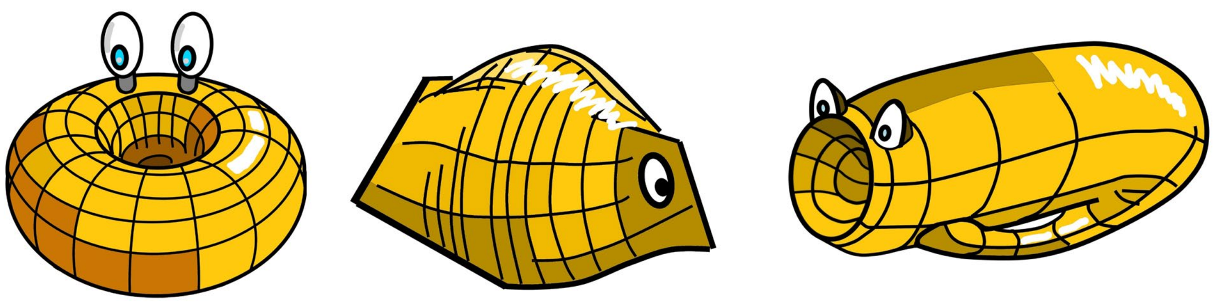

Figure 8: *Torus-swimmer, Pillow-fish and Klein bottle-hippo, representing the orbifolds.*

The two figurative graphics are inspired by tropical themes and at the same time highlight two special styles that emerge from some of the eight tilings found in Section 2. The first style is "wave-like": two tilings have viewed from a distance, a wave-like appearance, notably the 3^{d} tiling in Figure 4 and the 1st in 5. In the graphics, we added a sailing boat to the waves. The other style is the "connected-blocks", which appears from the 4th tiling in Figure 4 and the 2nd in 5. We turn that style into a patterned pineapple.

We used sublimation printing, which is done on special paper first. Subsequently, the ink is transferred by a heat press or a calendar machine onto the fabric. This technique has advantages over direct printing and silk printing: deep colors, easy up-cycling of used garments and one-piece production without fabric waste. We experimented with a variety of fabrics such as elastane, polyamide and cotton/polyester mixes. We also experimented with laser engraving and add-on 3D printing of thermochromic and glow-in-the-dark filament. Figure 10 shows some of the samples.

The garment construction is ready now; the main design drawing is in Figure 9. At the same time, we are exploring a few pied-de-poule orbifold-based fashion accessories (torus wrist band, twisted pillow bag, see the *supplementary materials* files). The garment design appears in an educational video for the TU/e Golden Ratio BOOST program.

Summary: the shirt conveys multiple mathematical stories about the enumerated tilings, yet appears tropical and playful. The combined engraving and sublimation printing add depth to the fabric, making the shirt stylish.

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Figure 9: *Shirt design. The shirt conveys multiple mathematical stories about the enumerated tilings, yet appears tropical and playful. It combines engraving and sublimation printing, showing all eight pied-de-poule tilings and three orbifolds.*

Figure 10: *Experimental print samples. Top row (from left to right): laser engraving with 3D printed TPU on top, engraving with 3D printed TPU to connect (engraved) patch, (small) sublimation print, monochrome sublimation print on pink jersey. Middle row: engraving and sublimation printing, engraving with 3D printed thermochromic filament. Lower row, from left to right: engraving with sublimation on top, engraving with partial sublimation on top, (small) engraving with sublimation on top.*

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