Helixation

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Abstract

In the list of names of the regular and semi-regular polyhedra we find many names that refer to a process. Examples are ‘truncated’ cube and ‘stellated’ dodecahedron. And Luca Pacioli named some of his polyhedral objects ‘elevations’. This kind of name-giving makes it easier to understand these polyhedra. We can imagine the model as a result of a transformation. And nowadays we are able to visualize this process by using the technique of animation. Here I will to introduce ‘helixation’ as a process to get a better understanding of the Poinsot polyhedra. In this paper I will limit myself to uniform polyhedra.

Introduction

Most of the Archimedean solids can be derived by cutting away parts of a Platonic solid. This operation , with which we can generate most of the semi-regular solids, is called truncation.

Figure 1: *Truncation of the cube.*

We can truncate the vertices of a polyhedron until the original faces of the polyhedron become regular again. In Figure 1 this process is shown starting with a cube. In the third object in the row, the vertices are truncated in such a way that the square faces of the cube are transformed to regular octagons. The resulting object is the Archimedean solid, the truncated cube. We can continue the process until we reach the fifth object in the row, which again is an Archimedean solid, the cuboctahedron. The whole process or can be represented as an animation. Also elevation and stellation can be described as processes. In this paper we will emphasize this approach to polyhedra, by first describing elevation and stellation, see [4], in this light and then define helixation and study the structure of Poinsot polyhedra.

Elevation - Stellation

The process of elevation can be described as follows: from each face of the polyhedron the mid-point is lifted, moved outwards, and is used as the top of a pyramid that can be formed on the edges we get when we connect the lifted point with each of the corner points of the original face, and on the edges of this face. In Pacioli’s definition we should stop the lifting process at the point where the distance between the lifted point and each of the vertices of the underlying face is equal to the edge length of this face. In the pictures of Figure 2 we see the pyramids growing on each of the faces of the octahedron. In the final state the pyramids are grown to a height in which the faces of the pyramids have become equilateral triangles. In total 8 x 3 extra triangles are "grown" on top of the faces of the original octahedron.

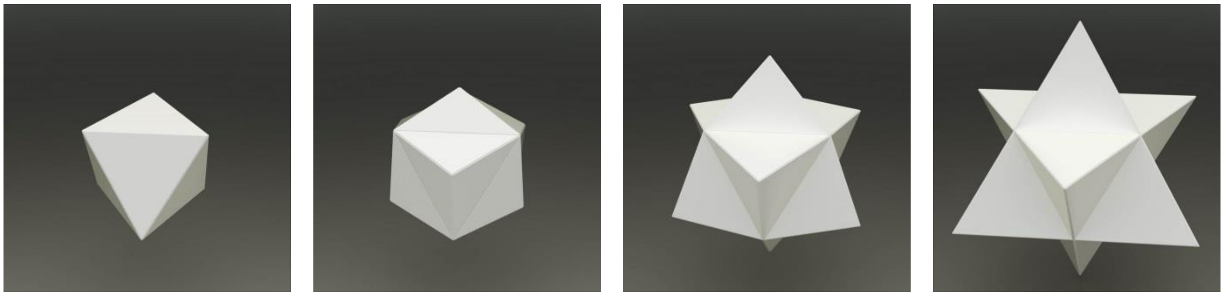


Figure 2: *The process of elevation. Stills of the animation.*

Leonardo's drawing of the object is shown in Figure 3a. Pacioli describes: "this object is built with eight three-sided pyramids, that can be seen with your eyes, and an octahedron inside, which you can only see by imagination." [3].
Stellation is the process of extending a polyhedron to form a new figure. Starting with an original figure, the process extends face planes, usually in a symmetrical way, until they meet each other again to form the closed boundary of a new figure. We can see a drawing of this structure by Wenzel Jamnitzer, a German artist from the renaissance, made in 1555 in Figure 3b. In his wireframe drawing we can see clearly that it is a combination of two tetrahedra. So two similar structures, but two different drawings.

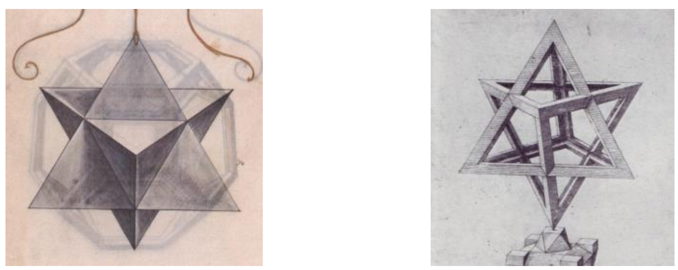


Figure 3: *(a) Leanardo da Vinci, "Octocedron Elevatus Solidus". (b) Wenzel Jamnitzer, "Stella Octangula".*

The difference between the two drawings is that Leonardo made a drawing of the elevation of the octahedral and Jamnitzer visualized the compound of two tetrahedra, or, what is later called, the stellation of the octahedron or 'Stella Octangula'. In 1619 Kepler defined stellation for polyhedra.

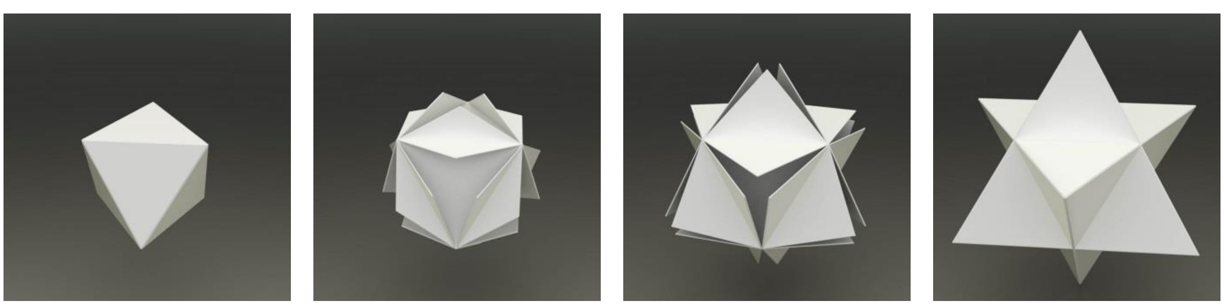


Figure 4: *The process of stellation. Stills of the animation.*

Figure 4 shows four steps of the stellation process applied on the octahedron. In the final state we see the stellation of the octahedron: the eight small triangle are “grown”, and have become eight big triangles, which together form the compound of two tetrahedra. The best way to see the difference between these two objects is by capturing the process in which they are obtained via animations.
Kepler’s two stellations of the dodecahedron can be described as regular twelve-faced polyhedra, formed with regular pentagonal star shaped faces. So instead of the convex pentagon, the pentagonal star is used as the shape of the face of the polyhedron. We can also describe a stellation as a process of extending the faces of the dodecahedron until they meet to form a new polyhedron. When we apply this process to the regular dodecahedron, we obtain the so-called small stellated dodecahedron. If we were to animate the process, we could see the convex pentagon gradually “growing” until it becomes the pentagonal star. Figure 5 shows four stills of the animation.

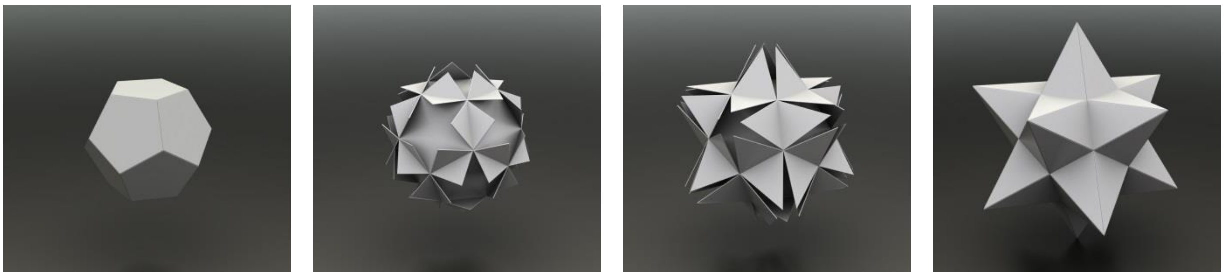


Figure 5: *The stellation process applied on the regular dodecahedron.*

There are two different regular polyhedra that can be made from twelve pentagonal stars. Kepler’s second polyhedron is the great stellated dodecahedron.

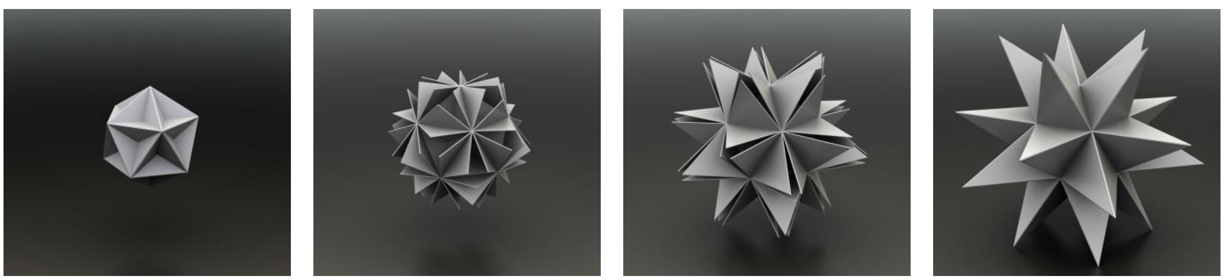


Figure 6: *The stellation process applied on the great dodecahedron.*

The great stellated dodecahedron is usually described as the result of a stellation process. The polyhedron we need then use as the starting point for the animation is the great dodecahedron, which is believed to be discovered by Poinsot, about 90 years later after Kepler’s discoveries. But already Jamnitzer published a drawing of this object in 1568 (Figure 7a) [2]. It can be difficult to determine the polyhedral structure of the great dodecahedron from the final stellation alone. We have already seen a good visualization of the small stellated dodecahedron in Escher’s print “Gravity” (Figure 7b). This method can also be used to make a clearer visualization of the great stellated dodecahedron (Figure 7c).

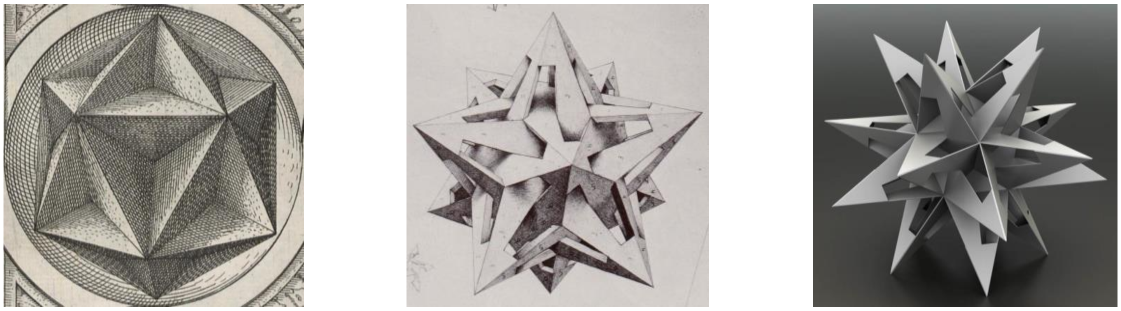


Figure 7: *(a) Jamnitzer's drawing of the great dodecahedron(b) Escher's drawing of Kepler's stellated dodecahedron. (c) The great stellated dodecahedron visualized using Escher's method.*

Visualizing the Poinsot Polyhedra

Next, we focus on visualizing the Poinsot Polyhedra, the great dodecahedron and the great icosahedron. We cannot be certain about the fact that Jamnitzer meant to draw a polyhedron built with twelve convex pentagons. It is one of his modifications of the icosahedron, and we cannot be sure which lines are edges and which lines are intersections. We need to be able to look inside to examine the real configuration.

Figure 8: *(a) great dodecahedron. (b) One face for the paper model. (c) Paper model of the great dodecahedron.*

Escher's method of cutting away parts of the faces, may seem to be the right way to visualize polyhedra in which faces intersect. We remove parts of the faces where the intersection takes place, in order to get a better view of the interior structure of the polyhedron. So a model of the great dodecahedron may look like the pictures of Figure 8a or 8c. In Figure 8b one pentagonal face with the cut out parts is shown.

Helixation

Another approach of visualizing the complex structure of the Poinsot polyhedra is the use of animation. In any case where we can interpret an object as a result of an operation applied on another object we can visualize this process with the technique of animation. We have seen this already with the operations truncation, elevation and stellation. Poinsot's polyhedra, the great dodecahedron and the great icosahedron, are built with exactly the same set of faces as the dodecahedron and the icosahedron. This fact brought me to the idea to look for a transformation of the dodecahedron to the great dodecahedron and for the icosahedron to the great icosahedron. The operation I found is the following: starting with the dodecahedron, we can translate the faces towards the center of the polyhedron and while translating we rotate the faces around their own midpoint. The total movement of each of the faces is a combined transformation, a screw displacement, or a rotary translation. We introduce the name helixation for the operation that applies the screw displacement of all of the faces until the edges of the faces meet other edges. So helixation is the operation that transforms one polyhedron into another polyhedron with the use of a screw displacement towards the center of each of the faces. Because the movement is a combination of rotation and translation we have 2 parameters that can be carried independently. The process of helixation of the dodecahedron is shown in Figure 9b-h. In Figure 9a we can see the path of a single face of the dodecahedron. The red line in Figure 9a shows the helical movement.

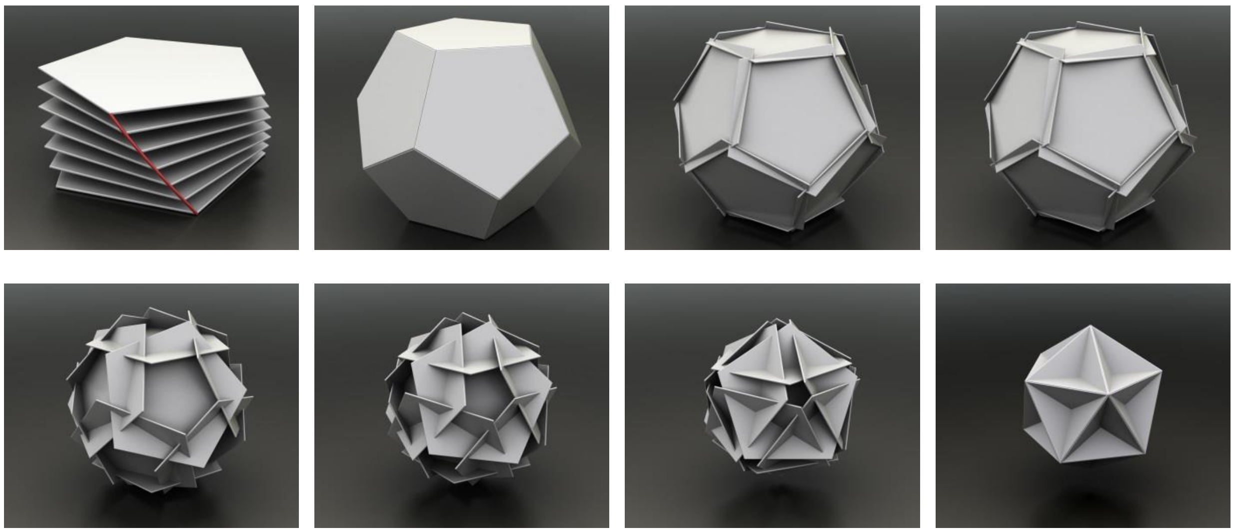


Figure 9: *(a) Helical movement of the pentagonal face. (b)-(h) Helixation of the dodecahedron to Poinsot's great dodecahedron.*

So now we have an operation which results in Poinsot's great dodecahedron and which can be shown by animation. We will examine the possibilities of the use of this method, helixation, by applying it to other polyhedra.

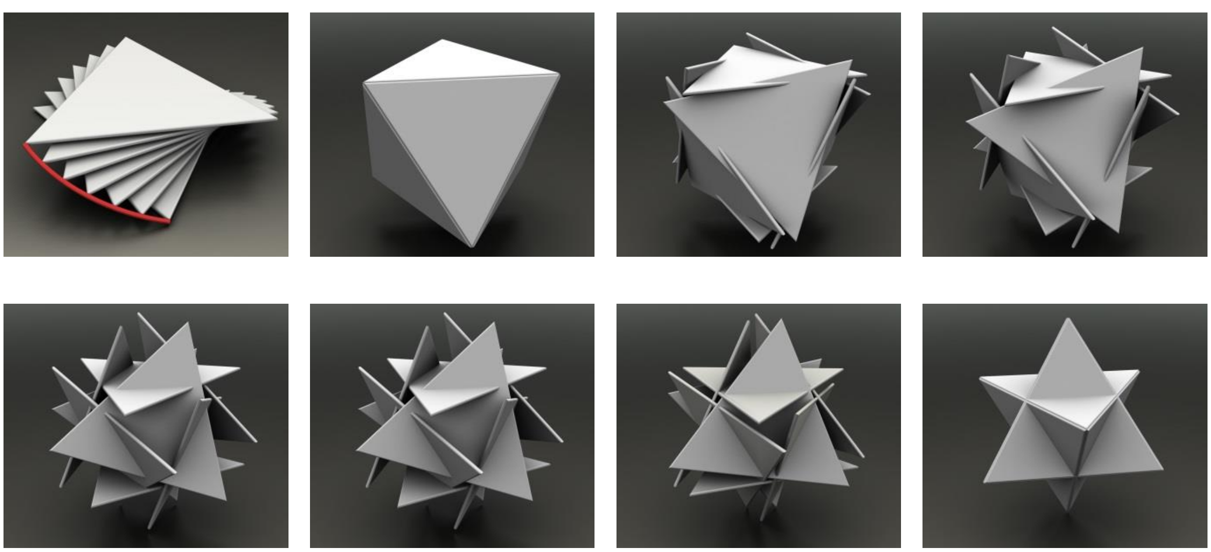


Figure 10: *(a) Helical movement of the triangular face. (b)-(h) Helixation of the octahedron to the Stella Octangula.*

The helixation of the octahedron is shown in Figure 10b-h. Figure 10a shows the helical movement of a triangular face. The resulting polyhedron, the Kepler star, is a compound of two tetrahedra. It may look that in this case helixation gives the same result as stellation. There is slight difference: helixation does not change the size of the faces.
Helixation of the cube doesn't give us a new polyhedron. This is also the case for the stellation of the cube. Also the helixation of the tetrahedron doesn't give us a new result. The final Platonic solid, the icosahedron, leads to Poinsot's great icosahedron, shown in Figure 11. The helical movement of one the triangular faces is shown in Figure 11a.

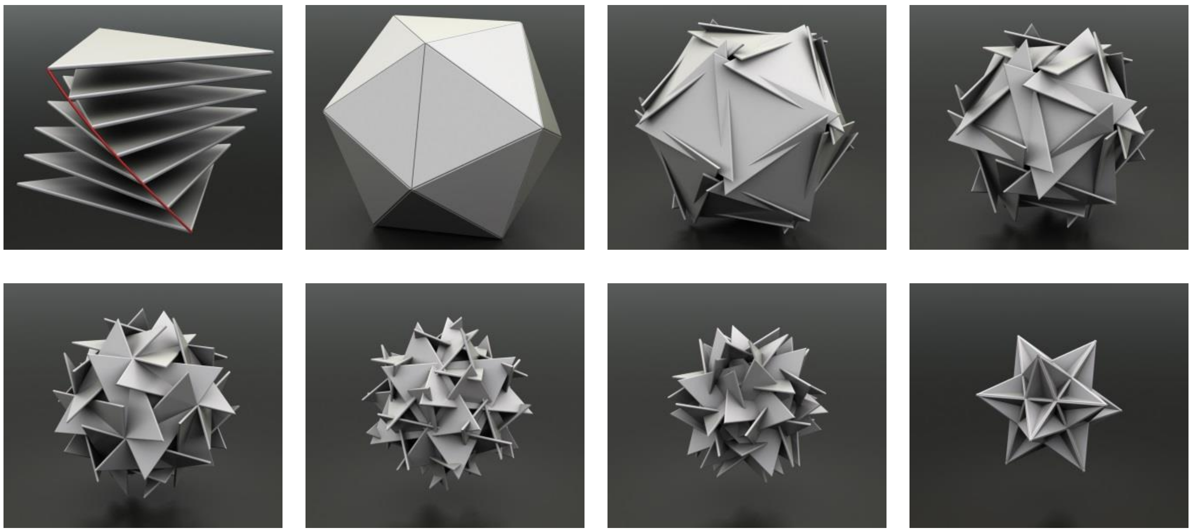
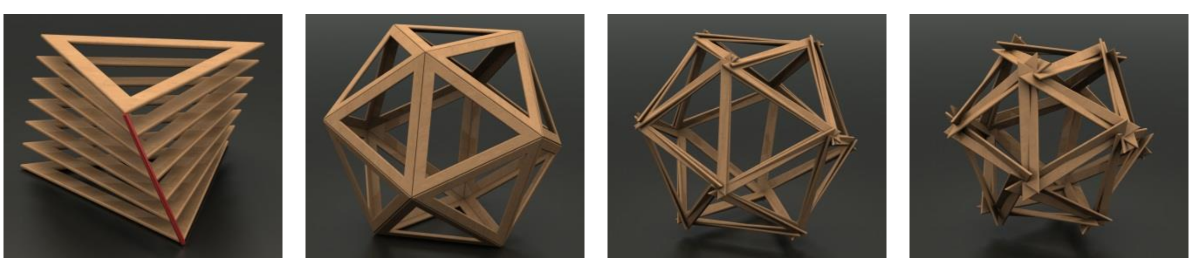


Figure 11: *(a) Helical movement of the triangular face. (b)-(h) Helixation of the icosahedron to the great icosahedron.*

Helixation is a combination of two different transformations: translation and rotation. So we need a certain distance for the translation and a certain angle for the rotation. These two parameters can be set separately, and so lead to different solutions. When we start moving the faces of the icosahedron towards the center we can use a different speed for the rotation. That we really can get other solutions is shown in Figure 12. With the combination of distance and angle shown in Figure 12a, the resulting polyhedron is not a single shape, but a compound of five tetrahedra.
So where we had the Stella Octangula, the compound of two tetrahedra, as an extra result of stellation, we now get the compound of five tetrahedra as an extra result of helixation. I used "open" triangles, as in Leonardo's 'polyhedra vacuus' drawings for the animation shown in Figure 12.



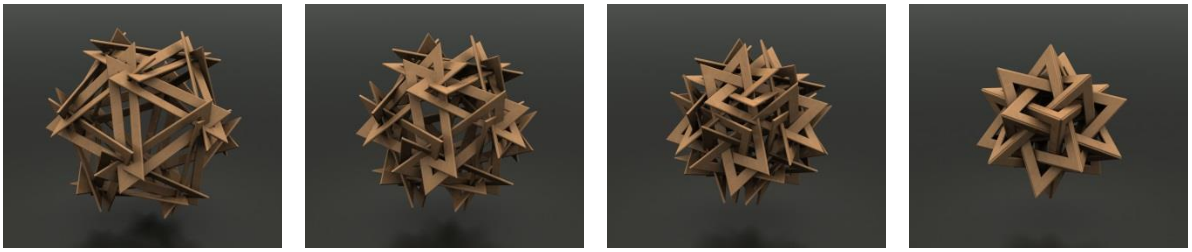


Figure 12: *(a) Helical movement of the open triangular face. (b)-(h) Helixation of the icosahedron to the compound of 5 tetrahedra, visualized using Leonardo's method.*

Helixation can also be applied to Archimedean solids. As an example we will take the icosidodecahedron as the starting polyhedron for the operation. A first result of the helixation of the icosidodecahedron is the uniform polyhedron great ditrigonal icosidodecahedron (Figure 13).

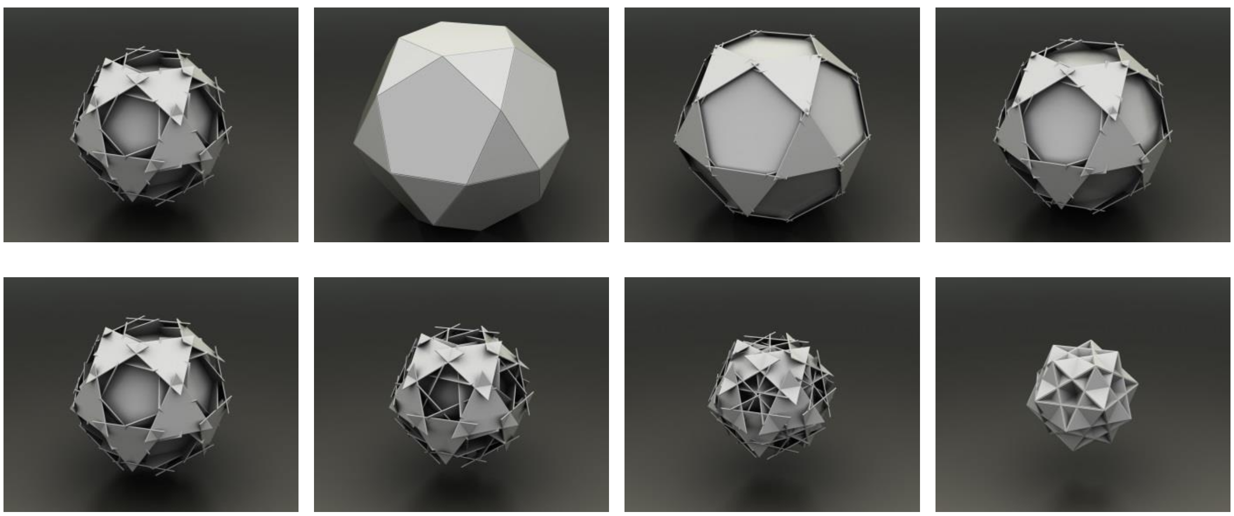
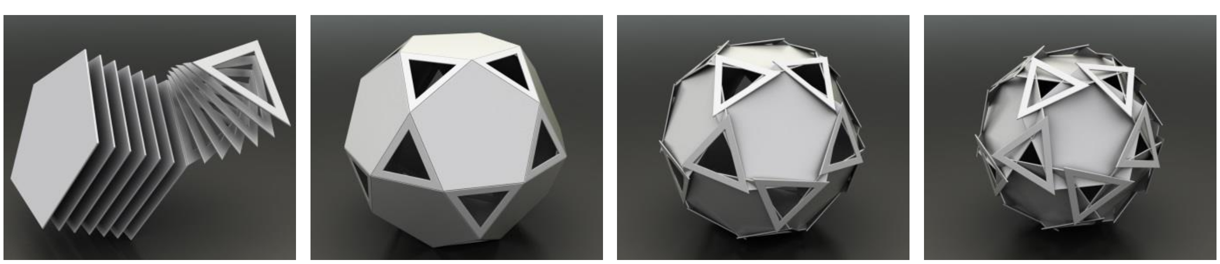


Figure 13: *Helixation of the icosidodecahedron to the great ditrigonal icosidodecahedron.*

For the helixation of the icosidodecahedron we have to move and rotate two different types of faces the triangular and the pentagonal faces. So we now have four different parameters. One other solution of the helixation of the icosidodecahedron is again a compound, the compound of icosahedron and dodecahedron. But there are more possibilities. In total there are seven different solutions of which five are compounds. The other solution which is not a compound is the helixation of the icosidodecahedron to Grünbaum's polyhedron 3,5,3,5,3,5,3,5,3,5 which he described in his paper "”New” uniform polyhedra" (2003) [1]. In Figure 14a the two different movements of both the triangular face and the pentagonal face are shown. The rotation angle for the pentagonal face is 0 degrees.



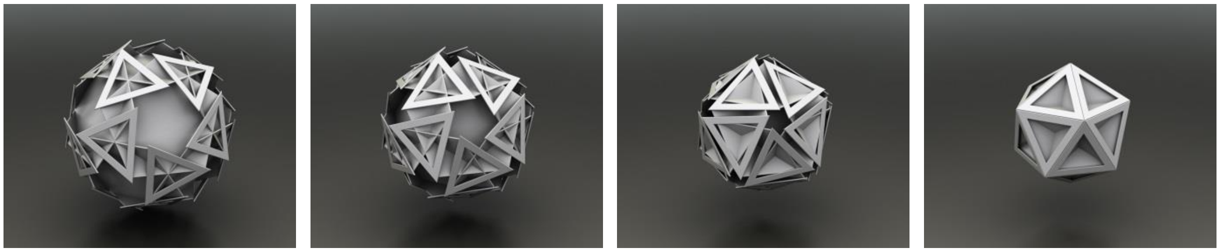


Figure 14: *(a) Helical movement of the pentagonal and the triangular open face. (b)-(h) Helixation of the icosahedron to Grünbaum's polyhedron 3,5,3,5,3,5,3,5,3,5.*

The result looks like a compound of the icosahedron and the great dodecahedron, but is a single uniform polyhedron. In his paper Grünbaum explains : “The vertices of many traditional uniform polyhedra determine regular polygons that are not faces of the polyhedron. In some instances, one can include such polygons and redefine adjacencies so as to obtain new uniform polyhedra.” ([1], page 6). This is shown, applied on the icosahedron, in Figure 15.

Figure 15: *(a) Visualization of Grünbaum's double polyhedron 3,5,3,5,3,5,3,5,3,5 using Escher's method. (b) Combined with Leonardo's method of using open triangles.*

Conclusion

Helixation seems to be a useful tool to explain the Poinsot polyhedra. It can also be used to examine and visualize other uniform polyhedra. Besides face-helixation, as presented in this paper, we can also define edge-helixation. With edge-helixation we rotate and translate the edges of the polyhedron towards the center of the polyhedron. With edge-helixation the number of edges stays the same. This leads to nice transformations of polyhedra with the same number of edges. For example duality of cube and octahedron can be illustrated with edge-helixation. Another example: all uniform polyhedra with 30 edges (dodecahedron, icosahedron, small stellated dodecahedron, great stellated dodecahedron, great dodecahedron, great icosahedron) as well as the compound of 5 tetrahedra can be transformed into one another.

References

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