Artistic Excursions with the Sierpinski Triangle

Roza Aceska[1] and Ben O'Brien[2]

[1]Mathematical Sciences, Ball State University, Muncie, IN; raceska@bsu.edu
[2]Virginia Commonwealth University, Richmond, VA; ben.obrien1895@gmail.com

Abstract

A course on the connections between mathematics and the arts was recently taught by the first author at the Honors College at Ball State University. The second author was a student in that course, who focused his final project on one particular fractal, the Sierpinski triangle. In this paper, we explore the properties of the Sierpinski triangle and discuss the creation of a Sierpinski star-polygon; we also present examples of digital artwork resulting from this discussion.

Introduction

Fractals [6] are objects generated by infinite repetitions, often paired with some type of zoom symmetry, though it is not necessarily the exactly same structure that is being preserved on all scales. Fractals can be found everywhere: their repeating shapes are observed in plant structures [6] as well as in African villages [1]. The Sierpinski triangle (Figure 1) has been used as a decorative element in 13th century Italian art [5].

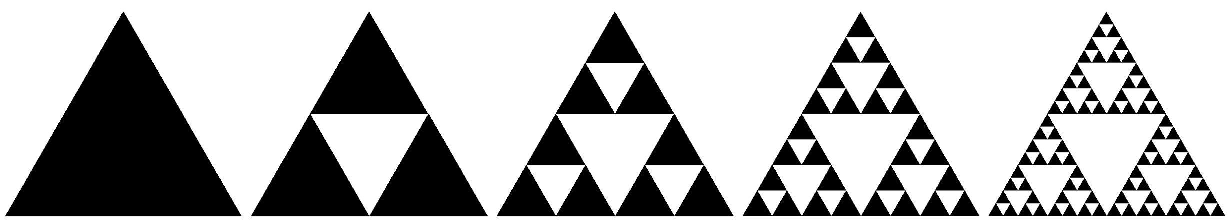


Figure 1: *Equilateral triangle and the first four iterations towards a Sierpinski triangle [11]*

The first author of this paper teaches a course on Mathematics and the Arts at the Honors College, Ball State University. This is a course envisioned as a research workshop for non-math majors, where students explore and re-discover various mathematics topics and create visualizations of math notions via ready-to-use code templates. Several weeks within the course are dedicated to the creation of a final project: students are free to explore math&art topics that mirror their interests and excite their intellect. Some students (including the second author of this paper) have created artistic math movies, and in the process they increased their familiarity with various mathematics topics. Within the course, students explore mathematics-and-arts topics from various sources, including the Bridges archive [4] and the AMS galleries [2], as well as various blogs on math, coding and art connections such as V. Matsko's [7]. Students also seek for math structures in the work of various artists. For instance, they analyze the types of symmetries and asymmetries of Mondrian and they explore the geometry of Hilma af Klint.

The Sierpinski Triangle for Maths and Arts Honors Students

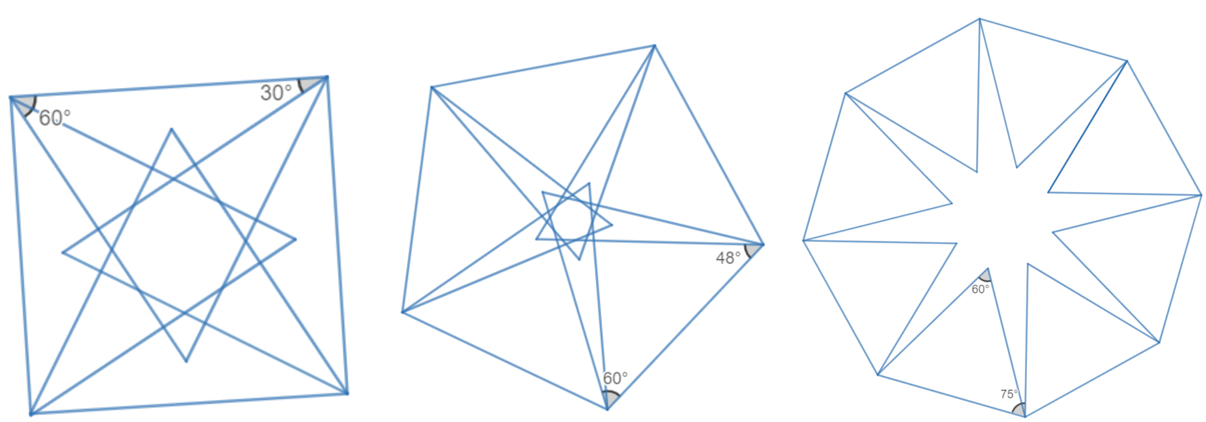
In the Math and Arts course, students are excited to learn more about the mathematics behind fractals, inspired by fractals' artistic visualizations of philosophical and mathematical concepts such as the concept of infinity or the concept of zero area. For instance, when studying the Sierpinski triangle (Figure 1), the students start with a hands-on exercise. They create the fractal by iteratively repeating a simple procedure: *Given an equilateral triangle, split it into four equilateral triangles, and remove the central triangle.*

Students are instructed to start working with an equilateral triangle of area 4 (each triangle after one iteration has area 1, then $1/4$ and so on). This is a good exercise because it gets them to anticipate the area of the Sierpinski triangle. After the students have brainstormed on what the area of the Sierpinski triangle may be, they proceed to calculate precisely the area of the removed triangles first. This quickly gets them to notice the fact that, by splitting the initial triangle into four triangles (each of area 1), and removing the central one, they have removed one triangle of area 1. Then, by splitting each of the three remaining triangles into four triangles of area $1/4$, and removing the central one from each, they have removed a region of total area $3/4$; and so on. They envision repeating the procedure infinitely many times, and realize they have removed a region of area $A = 1 + 3/4 + (3/4)^2 + \ldots + (3/4)^n + \ldots$, which is an infinite sum converging to 4. Thus, the Sierpinski triangle has zero area. In a similar manner, students find that the perimeter of all the remaining triangles in the Sierpinski triangle grows infinitely large.

The Sierpinski Star-Pentagon

The second author was a student in the Math and Arts course in Spring 2018; as his final project, he wanted to study the Sierpinski triangle, and create a *Sierpinski star-pentagon* (Figure 3) out of five (equilateral) Sierpinski triangles. Note that our creation, a pentagram within a pentagon, is different than the Sierpinski pentagons presented in [3, 10].

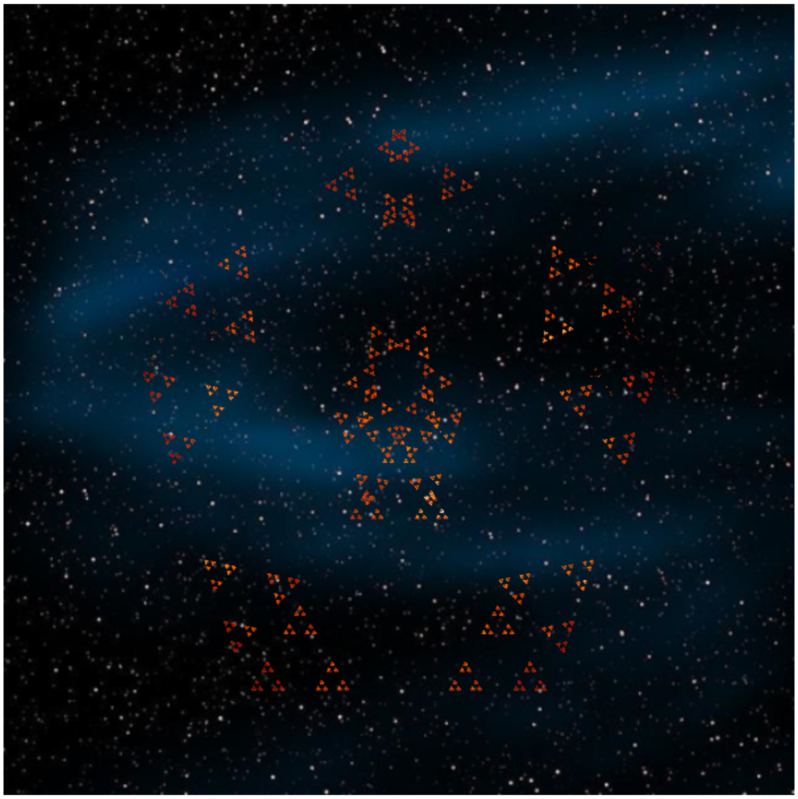
When working with plain equilateral triangles, we need five copies of the same equilateral triangle to create a pentagram within a pentagon (Figure 2b), which we call a *star-pentagon*. In fact, each *star-polygon* in Figure 2 is created out of n equilateral triangles, so that a star-like figure is enveloped into an n-polygon. If all n triangle copies were located at the same location in the plane, stacked one on top of the other, so that we only see the top triangle, an interesting geometry question opens: *What is the angle of relative rotation between two consecutive triangle copies that create a star-polygon?*



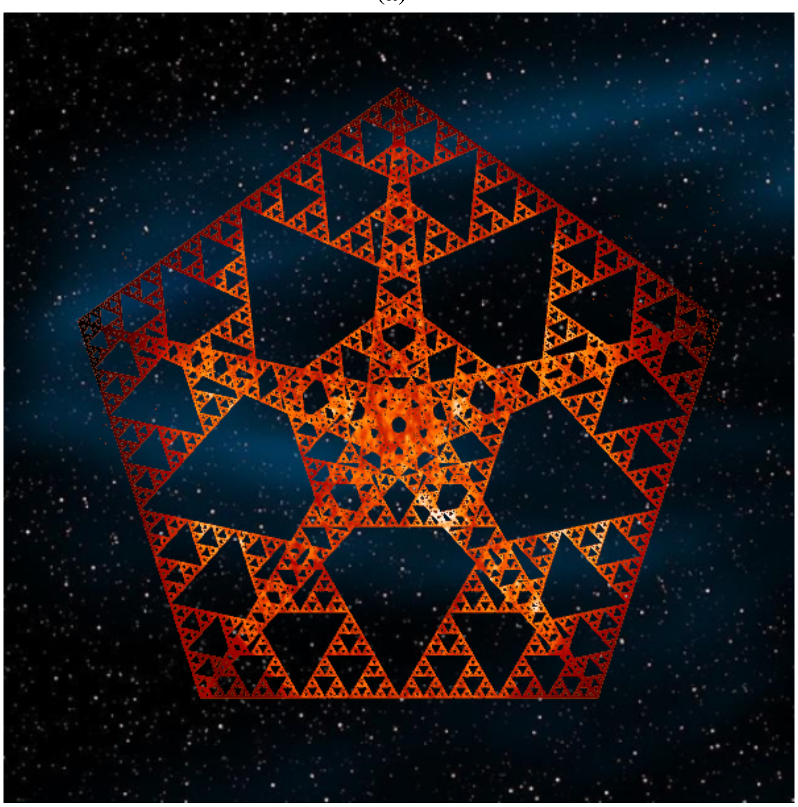
(a) *Four triangles* **(b)** *Five triangles* **(c)** *Eight triangles*

Figure 2: *Polygons created out of equilateral triangles - we used https://www.desmos.com/*

In general, the interior angle of a polygon with n sides is $(1 - 2/n)\pi$. If we want to build a polygon out of equilateral triangles, then the angle of overlap ($n < 6$) or gap ($n > 6$) between the two copies of an equilateral triangle needs to satisfy $\pi/3 + \pi/3 + x = (1 - 2/n)\pi$. Thus, $x = (n - 6)\pi/(3n)$. So, each new copy of the



(a)



(b)

Figure 3: *Selected frames from the Sierpinski star-pentagon movie [9] (a) Frame 0388.png, (b) Frame 0935.png.*

triangle needs to be rotated by $\pi/3 + x$, relative to the previous one.

In the case of the pentagon we have $x = -\pi/15 = -12°$, so the angle of rotation is $\pi/3 - \pi/15 = 48°$. That is, when creating copies of the Sierpinski triangle in Processing, each new copy of the original triangle needed to be rotated by 48°, relative to the previous copy.

The Artistic and Computational Side of the Project

The second author used Processing to create a short movie [9] of a starburst Sierpinski star-pentagon. The Python template (source [8]) he used, creates a right Sierpinski triangle; he altered the code so it produces an equilateral triangle, then created five identical copies, which are rotated and translated as needed. The side lengths of the triangles were manipulated to create a *starburst* effect, where the triangles grow and materialize into shape during the movie [9].

This idea of a starburst sparked another idea to make this piece an actual star set against the backdrop of space: the background was set to be a starfield. An image of the sun was used to plant orange RGB values: the program specifically searched out pixels with a specified color and replaced them with the corresponding pixels in the other image in memory, in this case, a sun (Figure 2).

Summary and Conclusions

We have presented math questions and artistic explorations related to the Sierpinski triangle, suitable for student final project work within a mathematics and arts course. Fractals are an intractable part of the artistic movie [9] created with the use of Processing: a tightly knitted group of infinite geometric figures representing just a small, infinitesimal speck in the near infinite universe.

Math aficionados and art lovers will enjoy guided journeys through the connections between mathematics and the arts within a college colloquium. Students can explore and discover (abstract) structures built from ideas, relations, patterns. They identify deep mathematical aspects found in artistic work, and beautiful art within mathematics, which inspires both a different point of view on art, and a different point of view on mathematics.

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