

Search for a Perfect Unequal Trapezohedron with Similar Faces

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Abstract

This paper searches for a perfect Trigonal Trapezohedron which has its lower 3 kite faces similar, but not congruent, to the upper three kite faces. The search results that there is no such 3-Trapezohedrons with convex faces. However one such polyhedron exists where its upper faces are concave kites (dart shaped) instead of a convex kite.

Introduction

3-Trapezohedron is a polyhedron formed by 6 kite shaped faces. This paper focuses on one such unequal 3-trapezohedron. Many unequal Trapezohedrons exists where the lower 3 faces are not similar to upper 3 faces. This paper answers the question that whether such polyhedron with its lower faces similar to upper faces exists. Equations are derived which satisfy the geometry to make both the kites similar. These equations are solved graphically to obtain the values of required variables. The values are used to construct the desired n-Trapezohedron.

Equations based on Geometry of 3-Trapezohedron.

Based on the geometry of 3-Trapezohedron equations in terms of its variables are derived.

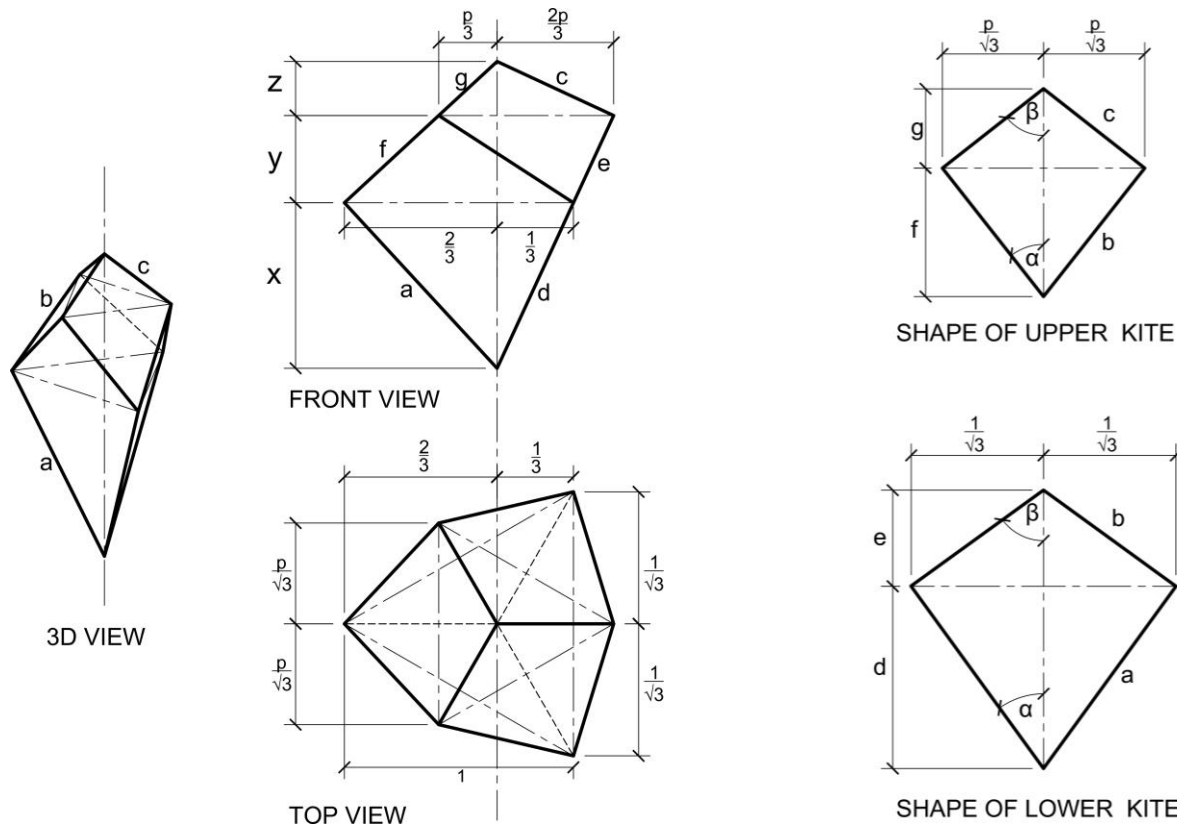


Figure 1: Views of a 3-trapezohedron.

Figure 2: True shape of upper and lower kites.

From Figure 1 by similarity of triangle following equations can be established:

$$3x = \frac{3(x+y)}{2p}, \therefore p = \frac{x+y}{2x}. \quad (1)$$

$$\frac{p}{3z} = \frac{2}{3(y+z)}, \therefore p = \frac{2z}{y+z}. \quad (2)$$

Equating equations 1 & 2, value of variable 'z' is obtained as follows:

$$z = \frac{y^2 + xy}{3x - y}. \quad (3)$$

By Pythagoras theorem following equations can be established:

$$a^2 = x^2 + \frac{4}{9}. \quad (4)$$

$$g^2 = z^2 + \frac{p^2}{9}. \quad (5)$$

$$b^2 = e^2 + \frac{1}{3}. \quad (6)$$

$$e^2 = y^2 + \left(\frac{2p-1}{3}\right)^2. \quad (7)$$

Conditions for Congruency of Bottom Angles

The angle ' α ' in Figure 2 of upper and lower kites should be congruent to ensure that both the faces of the kite are similar. Hence the following equation can be established:

$$\sin \alpha = \frac{1}{\sqrt{3}a} = \frac{p}{\sqrt{3}b}, \therefore b^2 = p^2 a^2.$$

Substituting for variable 'a' from equation 4, and for 'b' & 'e' from 6 & 7 respectively we get,

$$y^2 + \frac{4p^2}{9} - \frac{4p}{9} + \frac{1}{9} + \frac{1}{3} = p^2 x^2 + \frac{4p^2}{9}.$$

Substituting for variable 'p' from equation 1 we get the following:

$$y^2 - \frac{4}{9} \left(\frac{x+y}{2x}\right) + \frac{4}{9} = \left(\frac{x+y}{2x}\right)^2 x^2.$$

Simplifying this we get the following equation:

$$8x + 27xy^2 - 18x^2y - 9x^3 - 8y = 0. \quad (8)$$

Conditions for Congruency of Top Angles

The angle ' β ' in Figure 2 of upper and lower kites should also be congruent to ensure that both the faces of the kite are similar. Hence the following equation can be established:

$$\tan \beta = \frac{1}{\sqrt{3}e} = \frac{p}{\sqrt{3}g}, \therefore e^2 = \frac{g^2}{p^2}.$$

Substituting for variable 'e' from equation 7 and for variable 'g' from equation 5 we get the following:

$$y^2 + \frac{4p^2}{9} - \frac{4p}{9} + \frac{1}{9} = \frac{z^2}{p^2} + \frac{1}{9}.$$

Substituting for variable 'p' from equation 1 & variable 'z' from equation 3 we get the following:

$$y^2 + \frac{4}{9} \left(\frac{x+y}{2x}\right)^2 + \frac{4}{9} \left(\frac{x+y}{2x}\right) = \left(\frac{y^2 + xy}{3x - y}\right)^2 \left(\frac{2x}{x+y}\right)^2.$$

Simplifying this we get the following equation:

$$45x^4y^2 - 9x^4 - 54x^3y^3 + 6x^3y + 8x^2y^2 + 9x^2y^4 - 6xy^3 + y^4 = 0. \quad (9)$$

Graphically Solving Equations to find Variables of 3-Trapezohedron

Since the Trapezohedron should satisfy equations 8 and 9, they are solved graphically using online widgets on WolframAlpha [1].

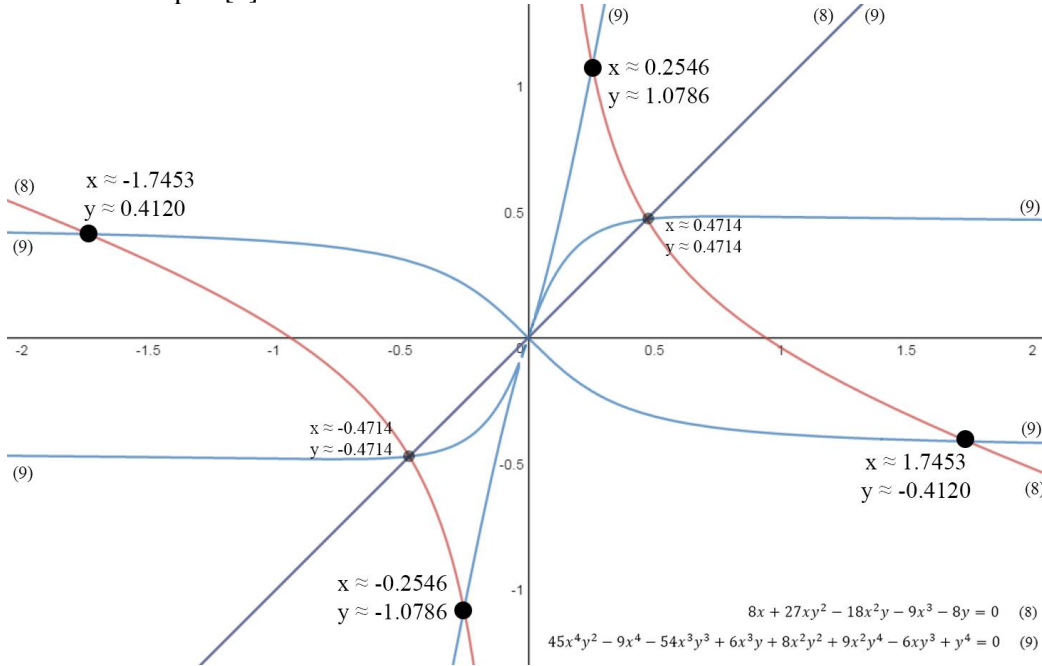


Figure 3: Plot of equations 8 & 9 showing the values of variables x & y at key intersection points.

Figure 3 is the plot of equations 8 & 9 showing the intersection points. Infinite intersection points exist along the line $y = x$, but since their values would not lead to an unequal Trapezohedron, they are discarded. Amongst other 6 intersection points, it is observed that two distinct polyhedrons can exist. When the variables $x \approx \pm 0.4714$ & $y \approx \pm 0.4714$ the polyhedron is a cube, which is trivial. All other four sets of ' x ' & ' y ' values lead to the formation of the desired polyhedron. The values in the first quadrant are used to find the other variables. A resultant negative value of ' z ' indicates that the shape of the upper faces is a concave kite (dart). The values of points in other 3 quadrants lead to the same shape.

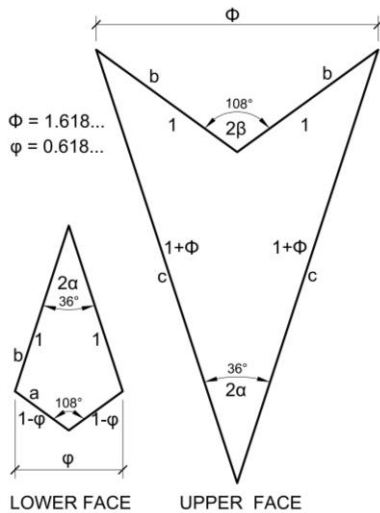


Figure 4: The two faces of Perfect 3-Trapezohedron

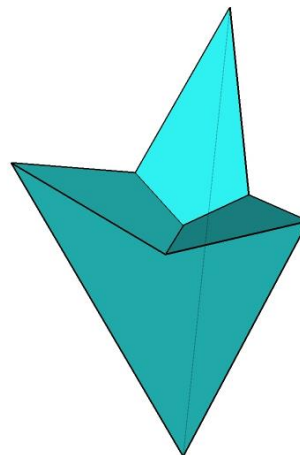


Figure 5: View of Perfect 3-Trapezohedron.

Figure 4 shows the geometry of the upper and lower faces of the Perfect 3-Trapezohedron. The presence of golden ratio in the sides is observed. While it is disappointing to see that the upper faces are concave rather than being a convex kite, similar to the lower faces, it is satisfying to note that if the upper faces are made into kite, similar to the lower kites, then 6 smaller and 6 larger kites can be used to construct a ‘Kite Dodecahedron’ (Figure 6). This module has all its interfacial angles constant and $\approx 116.6^\circ$, which is also the interfacial angle of a regular (Platonic) Dodecahedron. Hence the first law of crystallography holds good and it can be speculated that such crystals might exist in nature. It is also observed that two Perfect 3-Trapezohedrons and one such ‘Kite Dodecahedron’ are space filling modules as shown in Figure 7.

These new shapes can be used to make jewelry pendants, sculptures, gemstone cuts, hanging lights, vase, quartz crystals, Christmas tree decorations, paper ornaments, salt & pepper shaker, play blocks etc. They can also be used as designer product packaging and architectural space frames due to its space filling ability.

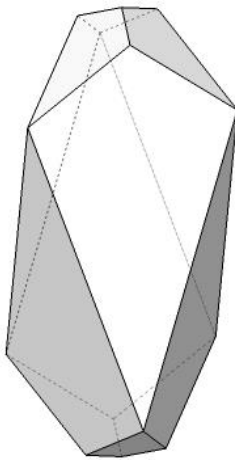


Figure 6: ‘Kite Dodecahedron’.

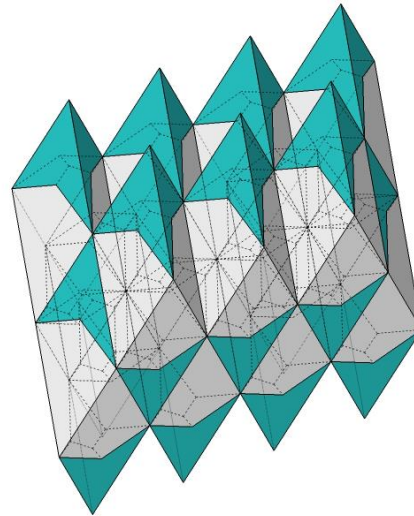


Figure 7: Geometry of space filling Perfect 3-Trapezohedron and ‘Kite Dodecahedron’ modules.

Perfect n-Trapezohedron where $n > 3$

Using the procedure described in this paper previously the Perfect 4-Trapezohedron can be analyzed. The two equations which need to be solved graphically are as follows:

$$x^3 + 2x^2y - xy^2 + y = 0 \quad (10)$$

$$-x^4 - 8x^3y^3 - 2xy^3 + 2x^3y + 4x^2y^4 + y^4 - 4x^4y^2 = 0 \quad (11)$$

A similar study is done for 5-Trapezohedron and the resultant equations are as follows:

$$[5xy^2 - 2y + (4 - 2\sqrt{5})x] (6 - 2\sqrt{5}) = 5x^3 + 10x^2y + 5xy^2 \quad (12)$$

$$(5x^2y^2 - x^2 + y^2) ((5 - 2\sqrt{5})x - y)^2 - 5x^4y^2(\sqrt{5} - 1)^2 = 0 \quad (13)$$

When the equations are solved in both the cases the result is that they have no real solutions except when $x = y = 0$. Thus it is concluded that no such Perfect 4-Trapezohedron and 5-Trapezohedron exists.

Conjecture

It is conjectured that a Perfect Unequal n-Trapezohedron does not exist, except when $n = 3$.

Reference

- [1] WolframAlpha, <https://www.wolframalpha.com>