Regular Whitney Decompositions

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Abstract

We use Whitney coverings for a region in the plane as a basis for creating pleasing visual imagery based on some simple geometric fractals that can then be used to create ornamental rugs.

In analysis, the Whitney decomposition of a planar domain is a partition of the domain consisting of squares with disjoint interiors of a size proportional to the distance from the boundary [2], as demonstrated in Figure 1.

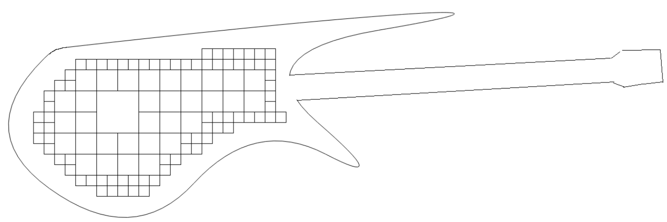

Figure 1: *Whitney covering of a "guitar".*

Usually, the squares have side length equal to a power of two and each side of any given a square touches one or two others (see Figure 1). However, even for simple domains it is difficult to quantify the pattern of neighboring squares within a Whitney decomposition of a domain, i.e., to describe which two squares of covering touch each other. In fact, this is possible only for special polygons. When studying many analytic properties of objects one can use a decomposition into other families of polygons instead of the family of squares. Each polygon from such family has to satisfy the following: the length of the shortest part of the perimeter between two of its points is uniformly comparable to the Euclidean distance between these points. This type of Whitney covering appears naturally in the study of conformal and quasiconformal mappings. In the present paper we show fractal domains and their Whitney-type coverings with extremely simple combinatorial structure; therefore we call them *regular Whitney decompositions*. We illustrate the domain construction process using the well-known von Koch's snowflake. Start with an equiangular triangle. At each stage we replace each edge of the figure with the polygonal curve obtained by replacing the middle third of the edge with two sides of the equiangular triangle having base equal to the removed middle piece.

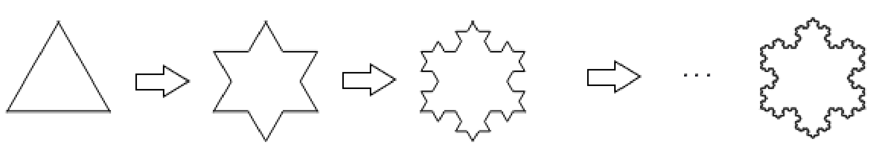

Figure 2: *Iterative construction of von Koch's snowflake*

Then we repeat the replacement process ad infinitum. The limiting object is a closed curve homeomorphic to a circle, called von Koch's curve. The region bounded by the von Koch's curve is called von Koch's snowflake.

During our study we discovered the regular Whitney covering of von Koch-type domains. For von Koch's snowflake, elements of covering are scaled copies of a "pants-shaped" dodecagon, scaled copies of a "palace-shaped" heptagon, and a single six-pointed star. The simplest way to describe our Whitney covering is to show it on a picture (Figure 3).

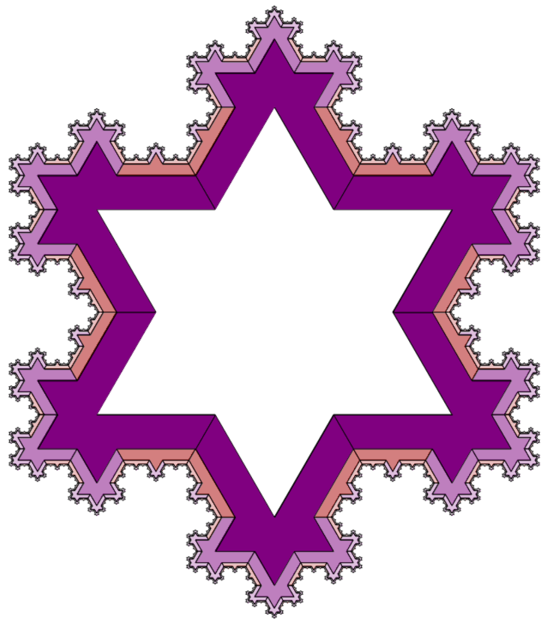

Figure 3: *Self similar Whitney decomposition of von Koch's snowflake*

The structure of this covering is easy to describe if we look at its *vicinity graph*. Its vertices correspond to the Whitney elements and two vertices are connected if corresponding elements touch each other. If we remove edges connecting vertices of the same size, we get the spanning tree T of this graph. We call it a *Whitney tree*. The root of T is a six pointed star with six "pants" shaped descendants. The type of a vertex describes direct descendants of this vertex (Figure 4).

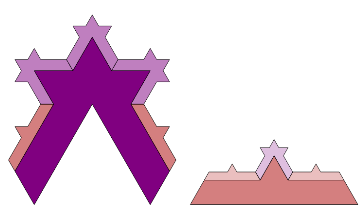

Figure 4: *On the left "pants" shaped polygon and its descendants, on the right "palace" shaped polygon and its descendants*

The descendants are similar to one of the polygons of the previous generation with a scale of $\frac{1}{3}$. It is easy to describe it's Whitney tree. We start from the root connected to six vertices. Then we draw the tree inductively preserving the rules of descendancy demonstrated in Figure 5. We construct the tree inductively. Starting with a root vertex representing the star-shaped figure at the center of the decomposition, we define six edges from the root vertex to six purple vertices representing the six "pants-shaped" polygons abutting the star. With each iteration, we then follow the rules of descendancy illustrated in Figure 5. So every purple vertex (representing the "pants-shaped" element) is connected to two orange vertices (representing the "palace-shaped" elements on each leg of the pants) and three purple vertices. Similarly, each orange vertex is connected to two orange vertices and a purple vertex.

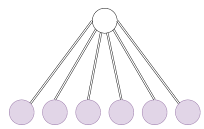

Figure 5: *Rules of descendancy for von Koch's snowflake.*

We found an astonishing aplication of this tree in the theory of Sobolev Spaces (see [1]). It is worth emphasizing that this type of covering is very particular and exceptional. We can construct such coverings only for a very special class of fractals due to the rigidity of the rules of the fractal construction. Moreover, it is not easy to describe how the shape of elements of the covering depends on the parameters of the fractal construction.

Another example of a domain which allows a regular covering was used by us to design a carpet. We start the construction with a square. Similar to the construction of the von Koch snowflake, we replace each edge at each iteration with a polygonal curve (Figure 6). We obtain

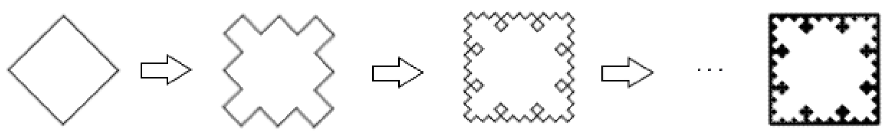

Figure 6: *Iterative construction of the second domain.*

Again, we repeat this procedure ad infinitum. The limiting set is not a simple loop this time, but a boundary of a domain whose closure fills a whole square. The corresponding Whitney covering is shown in Figure 8 and the descendance rules of are described in Figure 7.

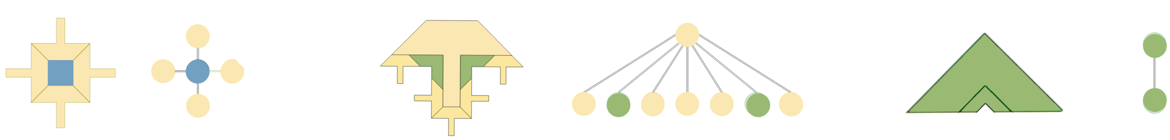

Figure 7: *Polygons with their descendants and corresponding rules of descendancy: central square (blue), octagon (yellow), hexagon (green).*

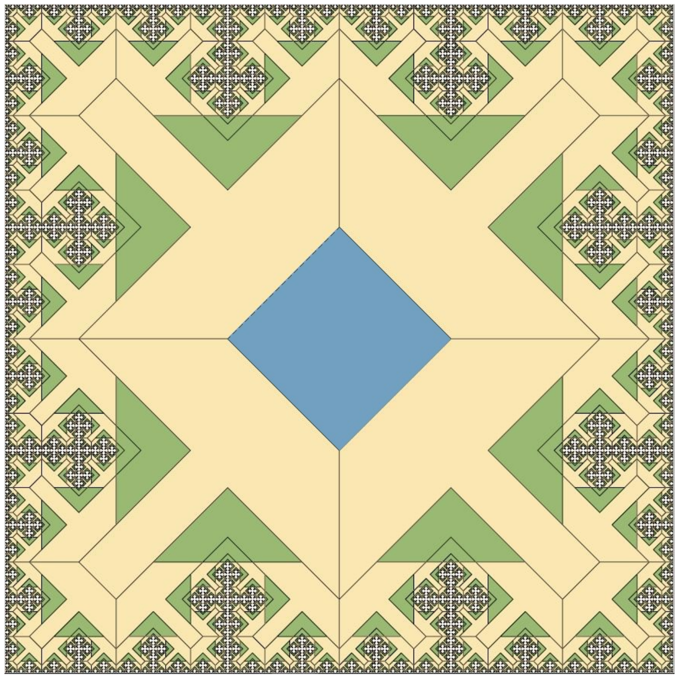

Figure 8: *Regular Whitney covering of the second domain.*

We used this decomposition as a starting point to define the geometrical pattern for our carpet. To create a more interesting visual effect, we filled the central square with a similar construction. And while geometry determined the pattern of the fractal domain, the choice of colors was left to the authors. We decided to follow the coloring scheme commonly found in traditional oriental rugs like the carpet pictured in Figure 9.

Figure 9: *Traditional oriental carpet and our design.*

References

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