Dissecting a Cube as a Teaching Strategy for Enhancing Students' Spatial Reasoning: Combining Physical and Digital Resources

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Abstract

This paper offers examples of how connecting concrete and abstract ideas through physical and digital modeling could enhance students' learning and creativity. We intend to emphasize the educational purposes of these examples and discuss some teaching possibilities in which mathematical and technological competencies of students are developed through physical and digital manipulatives, particularly designing a mathematical puzzle task digitally to be 3D printed. We outline examples for such experiments and share our observations on how these activities could improve students' geometric vocabulary and understanding of transformations. In addition to learning mathematics, we create opportunities for students to work collaboratively to develop a basic understanding of geometric modeling and to creatively solve a range of problems that naturally emerge during the digital to physical transition process. We believe that developing such learning environments and scenarios, and preparing teachers to be able to utilize alternative resources in their classrooms, could open new opportunities for learning geometry, enhance students' creativity and help to cope with technology challenges.

Introduction

The use of physical manipulatives in mathematics classes is usually motivated by their representational appeal in connecting concrete and abstract concepts (e.g. abacuses for numeral systems and basic operations, Cuisenaire rods for fractions, and Geoboards for geometry). Although we can find some examples in secondary schools, physical manipulatives still seem to be more associated with primary education, possibly due to their association with playfulness. Digital manipulatives, in turn, have steadily grown with the trend of the popularization of technologies in classrooms for different educational levels. Particularly for mathematics teaching, the use of Dynamic Geometric Systems (DGS) could support students in linking concrete and abstract concepts through their interaction with algebraic inputs and geometric constructions. In addition to that, mathematics syllabi suggest that by high school, students should be able to draw and construct representations of 2D and 3D geometric objects. This can be done using a variety of tools to solve design problems and understand the effects of simple transformations and their compositions [19][20]. Therefore, we started to reconsider the importance of designing tasks for educational purposes combining the modeling and exploratory aspects of DGS as well as to easily connect such outcomes with 3D printing to open students' and teachers' minds to new ideas and strategies in mathematical modeling and problem solving. In this paper, we present a perspective on integrating both physical and digital resources to explore the benefits of using each of them and how they can complement one another. Basically, the task involves asking students to develop their models digitally and convert these constructions to 3D printings. This activity also addresses the issues of adapting mathematical and design ideas, and how this process relies on, as well as how it nurtures, mathematical creativity through geometric modeling and students' multiple strategies to solve the tasks.

We aim to outline here a variety of ideas for students to develop solutions to this question:

How can we dissect a cube into n parts that have equal volumes and the same surface areas?

An immediate solution to this question is to cut the cube by n-1 planes parallel to one of the cube's faces giving all slices the same thickness. Besides this general solution for any integer n greater than one, it is possible to justify with some arguments of dragging points, sliding planes or compensating solids conveniently that the number of solutions is infinite. We hope that the interaction through some available digital models can guide the reader to realize that. The beauty of many problems in mathematics, nevertheless, lies exactly on this subtle contrast between knowing there is an infinite number of solutions and knowing how we can determine some of these solutions precisely. Moreover, sharing this kind of open-ended task makes students motivated in searching for their unique and personal solutions as for a well-hidden treasure, faced with the unlimited possibilities to solve the problem. It means we are also considering the differentiation in the sense that every single solution should be contemplated, regardless of which level from the spectrum of sophistication the solution belongs to, provided that it complies with the established conditions of the problem. Certainly, there are some known classical solutions for this problem and students should explore them as a starting point and then they may be inspired to create their own solutions. With this approach, we expect that students will start making mathematical conjectures through their experimental tasks and challenge themselves to find creative solutions. A classical example, which is usually explored in high school, is to *illustrate* the volume of pyramids, shown in Figure 1. The combination of both physical and digital resources can support students to develop their ideas and arguments in a complementary way. For instance, while the 3D printing solution could challenge students to assemble the cube as a puzzle, matching their edges and comparing their faces, the digital version offers a visualization of the transformation through the interaction with the software's features such as sliders, checkboxes, and color options, for better understanding. In this case, mathematical concepts such as symmetry, rotation or translation can be emphasized and better explained. Equivalent transformations can be also explored in the light of the spatial geometry, as shown in the last two pictures of Figure 1.

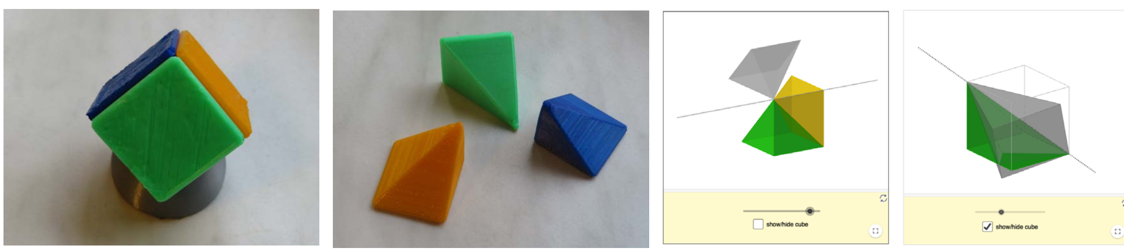


Figure 1: *Split cube in physical and digital versions*

We also highlight the relevance of bringing this kind of approach into schools as a possible connection to interpret particular solutions as master artworks, as shown by Rinus' solution in Figure 2, or even being inspired by such pieces to eventually generate new strategies and solutions. The interaction with the digital representation, in that case, again performs a meaningful role in outlining isometric transformations. Moreover, the geometrical construction can be a good challenge for tackling geometric properties, especially those regarding spatial reasoning such as the proper use of perpendicular lines and symmetry in 3D.

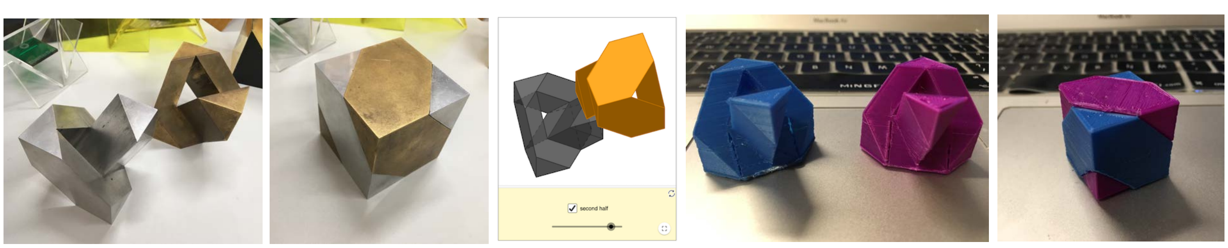


Figure 2: *An artistic solution from the mathematician and sculpturist Rinus Roelofs*

In the next sessions, we will present a set of alternative solutions for different values of *n* as well as some inspiring ideas from a variety of sources and the ways in which they can be adapted to inspire new ideas. All examples illustrated in this paper can be found in an online GeoGebra Book [21] to interact with or download and then adapt according to personal needs and interests.

Physical and Virtual Manipulatives

"In 2013, the National Council of Supervisors of Mathematics (NCSM) issued a position statement on the use of manipulatives in classroom instruction to improve students' achievement. In order to develop every student's mathematical proficiency, leaders and teachers must systematically integrate the use of concrete and virtual manipulatives into classroom instruction at all grade levels (NCSM, 2013). […]The history of manipulatives for formal mathematics teaching extends at least two hundred years. Such influences come from important figures in teaching including Maria Montessori (1870–1952), Jean Piaget (1896–1980), Zoltan Dienes (1916–2014), and Jerome Bruner (1915–2016) just to mention some. These innovators and researchers have emphasized the importance of authentic learning experiences and the use of physical tools as an important stage in development of learning and understanding. Piaget (1952) suggests that children begin to understand symbols and abstract concepts only after experiencing the ideas in a physical and concrete level. Dienes (1960) extended this to suggest that children whose mathematical learning is firmly grounded in manipulative experiences will be more likely to bridge the gap between reality in which they live and the abstract world of mathematics. Their pioneering work has inspired numerous initiatives and studies to explore manipulatives for student learning in mathematics." [10]

Connecting physical manipulatives either with games or creative designs could further enhance students' understanding of mathematics. On the one hand, to emphasize the opportunities that games and physical manipulatives could offer, Zoltan Dienes (2006), a pioneer of a worldwide physical manipulative movement, wrote that

"*My emphasis was on the use of mathematical games with appropriate learning aids (manipulatives), work, and communication in small groups with the teacher overseeing these groups…What I have been doing for over 50 years is not so much outside social issues but critical thinking about what mathematics is and what it can be used for and to have it presented as fun, as play, and in this sense it can be self-motivating because it is in itself a fun activity. I have critiqued mathematics being presented as boring repetitious activity as opposed to a way to think. So it is not so much critical thinking of social issues but as a way to train the mind […], understand patterns and relationships, in ways that are playful and fun."* ([18], p. 62-64).

On the other hand, Carlo Séquin [15] developed a set of design tasks with students in a graduate course on geometric modeling to promote the transition from 2D to 3D representations based on creative solutions. Although the dynamic aspect is not explicitly discussed in his paper, we can find a particularly similar approach in dissecting a sphere into equal parts rather than a cube.

As explained above, the use of manipulatives in the classroom has been advocated for a long time. In his work, Post [12] also brings perspectives from Lesh, Piaget, and Dienes, who emphasize the importance in applying physical materials with students, especially with younger ones. The increasing application of 3D printing in different areas led us to discuss possibilities for using it in educational settings creating, adapting, printing and exploring such manipulatives. Lieban et al. [8] discussed ideas and activities that make use of 3D printing in the school context. Some of the objects explored were logical games. They were developed in GeoGebra and Tinkercad, and then 3D printed. We chose these apps due to their intuitive interface for educational purposes and friendly connections to 3D printing.

Besides enhancing modeling competencies, these approaches with 3D printing can be a good opportunity to promote a student-centered learning environment. Students increase their motivation when working on activities that they create themselves [11]. In addition, moving from playing to making games, teachers and students develop skills that could go beyond mathematics, such as communication, creativity, collaboration and critical-thinking skills, the 4C's for 21st century demands [2][8].

Creating for Creativity in Math Environments

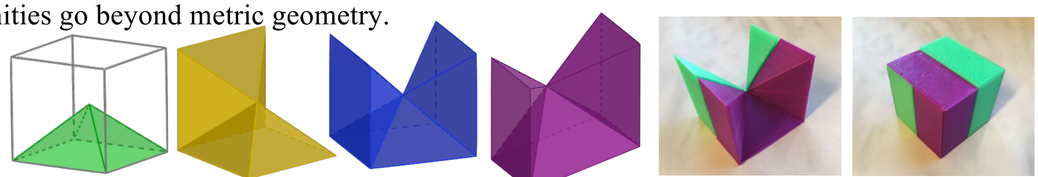
Our task design has its cornerstone in creative thinking for problem-solving. The strategies to achieve the goal of the task might arise from the interaction with either physical or digital manipulatives. We consider that providing students with some initial possible ideas can inspire them to create their own solutions. Recent workshops offered to teachers in the Czech Republic gave us some clues in that direction when in a group construction the participants realized that changes made by slight sliding a cross section generated a new solution. Creative thinking involves synthesizing skills, which depend on the abilities to combine parts to form a new whole using analogies, hypothesizing and planning a process [17]. For little adaptations from those models that already exist or even creating an original construction, it is also valuable that students have access to previous references to realize the limits and possibilities of the resources that they are working with. Working in collaborative teams and having brainstorms is also positive for promoting constructive debates and interchanging ideas [8]. Problem solvers beyond the classroom often are not isolated individuals, but rather teams of diverse specialists [4][5][13] who contribute with their perspectives based on their life experiences. Regardless of whether the problem solver is an individual or a group, researchers stress that the creative processes are a continuous cycle which tends to involve gradually sorting, clarifying, revising, refining, and integrating conceptual systems that are at intermediate stages of development [3]. Projects are more creative when the solution is redefined, revisited, and questioned numerous times during the process according to creativity scholars [7]. To support either technical or didactic debates regarding this or any other approach with 3D printing we created a GeoGebra Group [22] for promoting a collaborative discussion environment with developers, researchers, and users in general. There, participants can report their personal experiences and issues regarding the modeling process or final outcomes when using 3D printing.

Once students have a clear problem, in our case dissecting a cube into *n* parts with equal volume and surface area, and are involved or challenged by it, they feel more comfortable to formulate their own questions and search for strategies by themselves. This is one of the key premises in student-centered pedagogies such as project- or inquiry-based learning approaches. However, it is crucial that teachers offer alternatives in class to the standard textbook exercises as well as encouraging students to look around and observe how their surroundings can be related to the studied topic. Schoenfeld [14] argues that teaching methods that focus on standard textbook questions encourage the development of procedural knowledge that is of limited use in non-school situations. Although this practice has the value of consolidating technical skills, it tends to only slightly encourage open-ended questions and creative solutions for problem solving. Supporters of process-based work argue that if students are given open-ended, practical, and investigative work that requires them to make their own decisions, plan their own routes through tasks, choose methods, and apply their mathematical knowledge, students will benefit in a number of ways [1]. Accordingly, we consider the relevance of presenting some references that can work as starting point for students' ideas, especially when they look familiar to them. Later, we will present two solutions for our question inspired by different sources, one that comes from a game and the other one from a structure presented at the previous Bridges Conference in Stockholm.

Creativity involves divergent thinking, what some enthusiasts call *"thinking outside of the box"* and convergent thinking, or the ability to have analytic focus into the solution based on the stages of the cycle previously listed [7][17]. Mednick [9], a psychologist who was one of the earliest to propose processes for creative thinking, drew evidence from poets as well as mathematicians and scientists to theorize that creative "performances" of them were due to the unique "combinations of associative elements". On the subject of mathematical creativity, researchers address theoretical foundations and empirical studies to investigate how creativity could affect an individual's mathematical ability or, conversely, if mathematical knowledge may enhance mathematical creativity [6].

Dissecting a cube into *n* parts

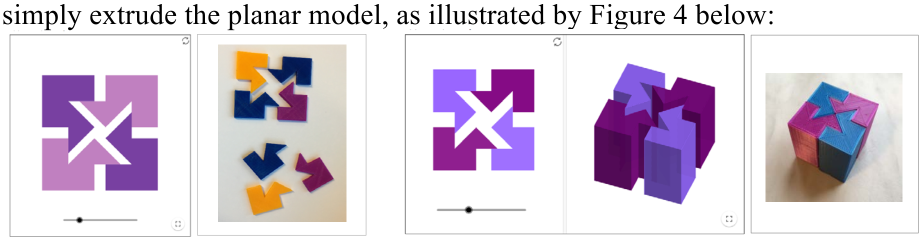
While designing the pieces of the cube to fulfill the conditions of the task, a natural process is to start by adapting solutions that are already known to students in order to invent new models. Creative thinking is a dynamic, exploratory process involving constant questioning and reshaping of the problem and solution [7]. For that reason, it is important to feed students a range of samples that work as a starting point of a brainstorming session, where they can discuss alternatives and restrictions to develop their personal ideas or puzzles. The next example, for instance, shows how we can combine a standard pyramid (Figure 3 (a)) in different ways to obtain new solutions. Once the initial model represents 1/6 of the cube, joining two or three of them makes up 1/3 and 1/2 of the cube as shown in Figure 3 (b) and (c). A slightly less intuitive solution can be obtained by continuing in the dissection process as illustrated by Figure 3 (d)-(e)-(f). In that case it is important to be convinced why the volume and the surface area are still the same among all parts. Observing the symmetry and fractions involved we realize that the mathematics learning opportunities go beyond metric geometry.



(a) (b) (c) (d) (e) (f)

Figure 3: *Developing different solutions from the same basic shape*

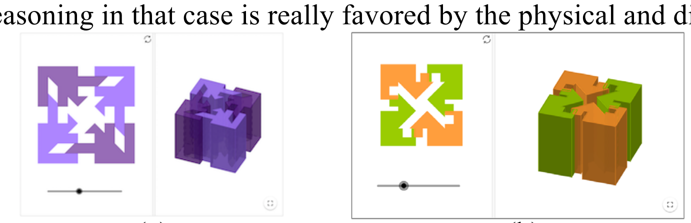
This task can offer a good opportunity to extend ideas from the plane to space as well as to open discussions for those ideas that cannot be directly transferred in general. In the *Discussion* session, we outline some other results in math and their analogues comparing both 2D and 3D spaces. As a particular example we use a solution obtained initially in 2D to split a square into four pieces with equal perimeter and surface area. We observe similar conditions in both 2D and 3D spaces: that is, dividing a square in such a way is equivalent to the splitting of a cube into four pieces that have the same area and volume when we simply extrude the planar model, as illustrated by Figure 4 below:



(a) (b)

Figure 4: *From 2D to 3D space, analogies that are transferred by extrusion*

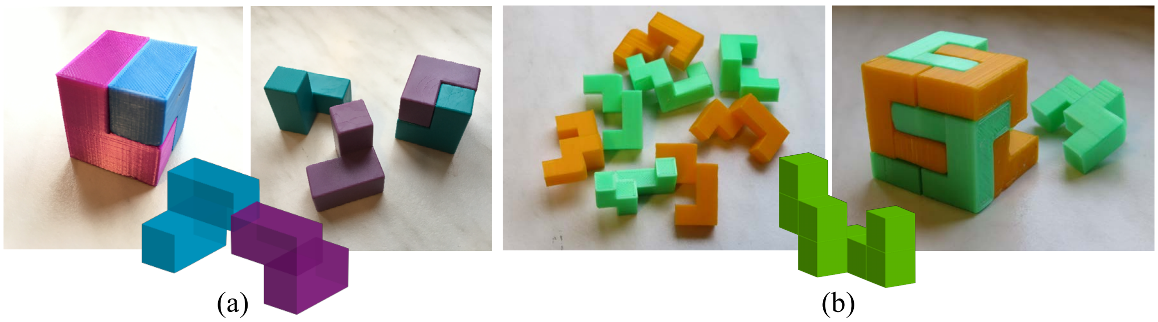
Regarding these physical and digital explorations, we observe some remarkable features. While the physical models allow more freedom in testing and assembling the cube, the preset mode to assemble it in the digital version can highlight certain regularities or particular behaviors in dragging all parts synchronized to fix them as a single final piece. As a particular example we show in Figure 5 how two different models adapted from the previous ones (Figure 4) behave with respect to the convergent movement from the corner to the center. The pattern presented in Figure 5 (a) surprisingly could be assembled without taking the pieces out of the same base plane. Nevertheless, in the physical version of the model represented in Figure 5 (b) we have to take at least one piece out from a standard plane in order to fix the cube. The digital version shows the overlapping when repeating the same movement as the previous one. Spatial reasoning in that case is really favored by the physical and digital integration.



(a) (b)

Figure 5: *Adaptations from the previous model*

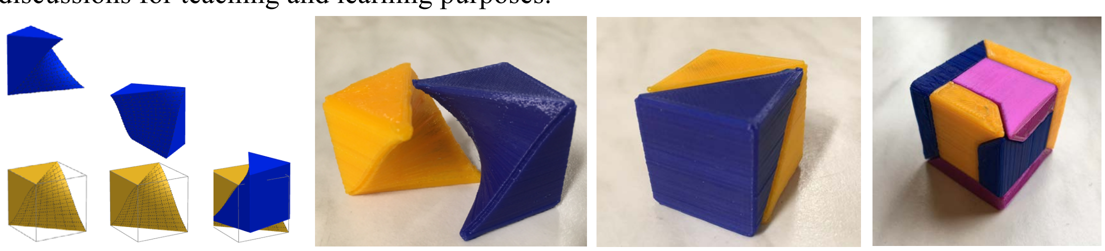
Another perspective in using the kind of approach illustrated in the previous examples is the introduction of sequence or recursive thinking. Particularly, Figure 5 (a) has the same strategy of construction as Figure 4 (a), but with more successive steps. Such ideas can open a good discussion about series and infinity. Questions such as *"Comparing two strategies for dividing a unit cube into the same number of pieces, which of them minimizes the surface area?"* launch the students into an inquiry-based learning task and can bring them to a link with another important mathematical field, for instance, the study of minimal surfaces. Nevertheless, a more natural way to start is from basic cubes or familiar shapes. The inspiration could center around, for example, the classical *Rubik's Cube*, which is quite popular among kids and teenagers. In Figure 6 (a) we present an idea from a board game (PUEBLO) and in Figure 6 (b) clusters of cubes were used as a strategy to split the cube in 8 equal parts. This idea was presented in the last Bridges Conference [16].



(a) (b)

Figure 6: *Split cube in physical and digital versions, inspired by the game PUEBLO and an artwork from the previous Bridges Conference*

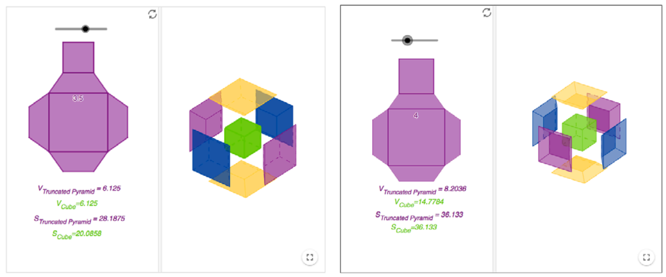
An alternative solution (Figure 7(a)), adapted to GeoGebra by Steve Phelps [23], shows a dissection of a cube into two equal parts and it works as if the cube was entirely split by the action of a screw. The digital representation for that in DGS clearly represents the rotational movement in 3D. It is interesting to note that the same strategy can be also used for other shapes besides cubes such as spheres, cones, and tetrahedrons. Some other inspiring models are also shared in a range of other collaborative platforms as well, such as Thingiverse, for instance, where we found the piece files for the cube shown in Figure 7 (b). Such platforms work as a great repository or source of inspiration. However, this last model does not completely satisfy the conditions of the initial problem since there is a small empty space inside to allow all the pieces to fit together. The exploration of this model in GeoGebra, from its "connecting-points construction" and through the use of opacity control feature, can be helpful to highlight such gaps and to open discussions for teaching and learning purposes.



(a) (b)

Figure 7: *Rotational movement in 3D explicit in the digital version (a) and a creative model shared in the collaborative platform Thingiverse (b)*

Last, but not least, we want to finish this collection of examples with one that could be really applied as an open-ended challenge and suggesting how some ideas can be adapted and improved. We offer two examples that represent a cube split into 7 parts, however, they do not completely fulfill the conditions of the problem, since in Figure 8 (a) all pieces have the same volume but not the same area, while in Figure 8 (b) all pieces have the same area, but not the same volume. The idea is to discuss whether or not is possible to obtain certain divisions with the same area and the same volume. Can these examples help? How? Which one would be better in that case? Why? These kinds of questions emphasize the importance of the role of the teacher in this approach, guiding the questions in a way that supports students in their own discoveries. Naturally, it is important to be familiar with the problems in advance. Another important aspect in this example is showing that the pieces do not necessarily have to be all the same to fulfill the conditions of the problem as in all the previous examples. A similar strategy could be used to split a cube into five or nine parts with equal volumes and surfaces areas.



(a) (b)

Figure 8: *Two examples to motivate a proper solution based on them*

Discussion

Several problems in mathematics have a certain correspondence between their versions in 2D and 3D spaces. Many times, comprehending them in one of these representation first is helpful to better visualize or comprehend the other representation. Usually, it is more common that this understanding step occurs from 2D to 3D space, but this is not always the case. This kind of activity helps to strengthen this connection and the development of new 3D features for DGS have contributed a lot to this process. Next, we present some examples that highlight this idea. One of the of the most well-known theorems, the Pythagorean Theorem, also has an "equivalent version" in 3D, although it is generally ignored. If we consider a right pyramid (like a corner of cube dissected by a plane not parallel to any of its faces), the theorem is also valid; in this case, it means the sum of the squares of the surface areas of the faces that merge in the corner is equal to the square of the surface area of the face that is opposite to the same corner. The Varignon Theorem is also true for skew quadrilaterals even though there are only few mentions about that in the literature. In February 2018, GeoGebra Materials consisted of 175 worksheets related to Varignon's Theorem, of which only three of them mentioned the result is also valid on 3D space. Viviani's Theorem has also its analogous version in 3D. The Monsky Theorem, in turn, says it is not possible to dissect a square into an odd number of equal area triangles. If we consider the cube as a natural extension of the square to 3D space and the pyramid as the 3D version of a triangle in the sense that we have a unique base and a merging vertix out of this base, then the equivalent of this theorem clearly fails, as illustrated in Figure 1.

Conclusion

In this paper, we outlined examples and experiences through developing environments and activities for connecting concrete and abstract ideas, physical and digital modeling, with a mathematical puzzle task, and manipulatives. These initial ideas could serve as examples and motivations to further develop such resources, pedagogies and practices, and encourage colleagues to carry out research in this area. At Bridges, we will offer further details of our work and show the actual developed resources and hope to develop connections with like-minded colleagues to further mathematics teaching and learning in this possible new direction.

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