Which Flower is It?

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Abstract

For any positive irrational number ω, the arithmetic ω flower is $\mathcal{F}_\omega = \{n(\cos 2\pi\omega n,\, \sin 2\pi\omega n) \,|\, n \geq 0,\, n \in \mathbb{Z}\}$. When $\omega = \phi$, the golden mean, and the scaling factor n is replaced with $\sqrt{n}$, it is well-known that the seed arrangement in an ideal sunflower is modeled by the points in $\mathcal{F}_\phi$—and that the seeds appear to sort themselves into a family of f_n radially symmetric spirals, where f_n is the n^{th} Fibonacci number, for any $n \geq 1$. A similar phenomenon occurs for ω in general; and we outline how to construct $\mathcal{F}_\omega$ of multiple coronas consisting of successively larger numbers of petals q_j corresponding to fractions $\frac{p_j}{q_j}$ that are good approximations to ω, as integer j increases. Lastly, given a flower $\mathcal{F}_\omega$ where ω is unknown, we reverse engineer the process and recover the good approximations for ω that are implicitly evident within the flower.

The flower bouquet of Figure 1 represents the irrational numbers e, ϕ, and π in some order, where ϕ is the golden mean $\phi = \frac{1+\sqrt{5}}{2}$ and e is the natural number. Which flower is which? The number of petals in the successive corona layers for each flower are clues. The answer appears at the end of Example 3.

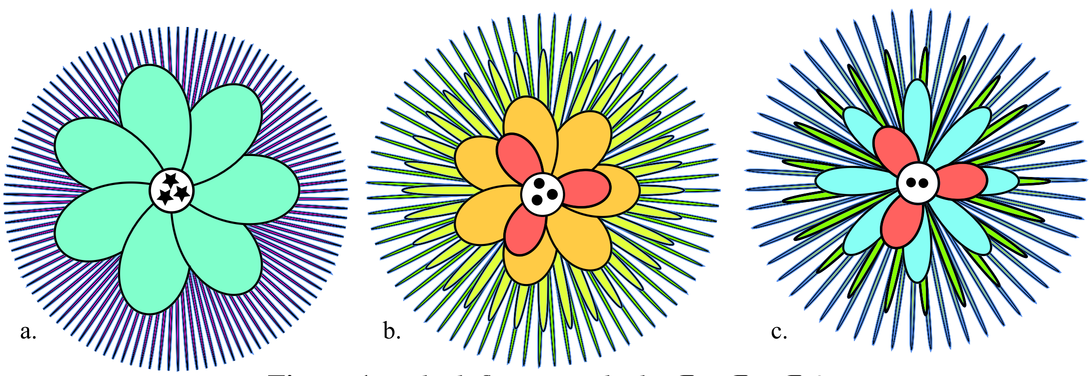


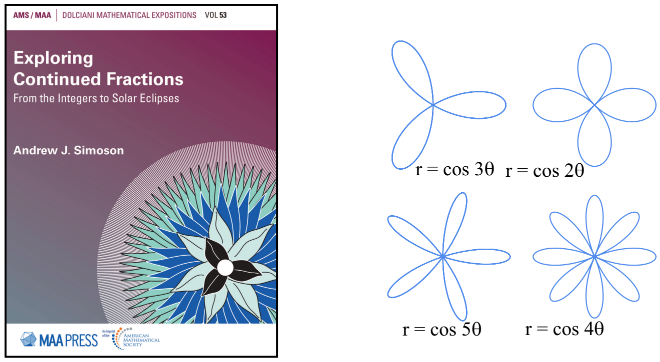
Figure 1: *Which flower is which:* $\mathcal{F}_\phi$, $\mathcal{F}_\pi$, $\mathcal{F}_e$?

In this paper, given any positive irrational number ω we outline a way to a construct a flower, denoted $\mathcal{F}_\omega$, that characterizes ω. We then show how, given the graphical flower of an unknown ω, we can reverse engineer the process and thereby approximate the value of ω.

Toward a stylized flower model

Not long ago a book cover design editor asked if I had a cover art idea for my forthcoming book on continued fractions [8]. *How about a stylized flower?* I suggested.

As we will see, a flower of layered coronas is the very embodiment of a continued fraction algorithm. For a given positive irrational number ω, a continued fraction algorithm generates fractions $C_n = \frac{p_n}{q_n}$, called *convergents*, that approximate ω, where p_n and q_n are relatively prime positive integers, n is a nonnegative integer, q_n increases as n increases, and p_n is the integer nearest $q_n\omega$, denoted $p_n = [q_n\omega]$. If you understand the construction of the flowers of Figure 1, I claim that you understand continued fractions.



a. A book cover. b. A few polar flowers.

Figure 2 : *Flowers, flowers everywhere.*

In due course, I sent the editor the flower design of Figure 2.a which was then positioned to run off the cover for added artistic effect. Where is the mathematics in such flowers? The answer is two-fold: (i) The number q of petals in any corona of $\mathcal{F}_\omega$ is the denominator of a fraction well-approximating ω, and (ii) Given a wheel freely rotating in place at ω revolutions per second and whose radius increases uniformly in time, tracking a tack P on the wheel's rim at unit second increments allows us to perceive the collection of P's positions as a family of q spirals. We formalize (i) in Theorem 1 and (ii) in Theorem 4.

To begin, here is one way to construct a flower of n petals, a *corona*, for any given positive integer $n \geq 1$. When $n = 1$, a single circular disk might be best, such as exemplified in a bellcap mushroom. When $n \geq 2$, a simple approach is to use rotational symmetry and construct a wagon wheel skeletal template of n uniformly spaced spokes—a stick-like figure. To embody this skeleton with petals, we could simply use the polar flowers $r = \cos n\theta$, as illustrated in Figure 2.b. But such an arrangement generates spindly-shaped coronas. Besides that flaw, the polar flower $r = \cos 2n\theta$ has $4n$ petals, not $2n$. A more luxurious arrangement results if we use n somewhat overlapping copies, symmetrically arranged about a hub, of a single petal from the polar flower $r = \cos m\theta$ where m is less than n, a trick we used in generating the first few corona layers in each of the flowers of Figure 1. As further embellishment we could give each petal veins and serrated or multi-lobed edges as is done in Figure 6.b.

In sports we award medals for excellence. In somewhat the same way let us award bronze, silver, and gold designations to those reduced fractions $\frac{p}{q}$ that approximate ω in the following respective ways:

$$\text{bronze: } \left|\omega - \frac{p}{q}\right| < \frac{1}{q^2}, \qquad \text{silver: } \left|\omega - \frac{p}{q}\right| < \frac{1}{2q^2}, \qquad \text{gold: } \left|\omega - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

As a note of historical interest, in 1891, Adolf Hurwitz showed that for any real number ϵ, $0 < \epsilon < 1$, there are some irrational numbers ω for which only a finite number of fractions $\frac{p}{q}$ exist for which $|\omega - \frac{p}{q}| < \frac{\epsilon}{\sqrt{5}q^2}$; a slick proof for this result appears in Ford [3, pp. 94–95]. So for all practical purposes, no platinum medals will be awarded.

To continue with our flower construction, given ω, we seek an increasing sequence q_n for which $C_n = \frac{p_n}{q_n}$ (with $p_n = [q_n\omega]$) are at least silver medaling fractions. Then we layer successively larger circlets of q_n symmetrically-arranged petals about a given point, $1 \leq n \leq k$, for some integer k. Finally—to aid in reverse engineering the flower (an element I had neglected to include on the book cover of Figure 2.a)—we embed a total of $[\omega]$ dots within the flower's center, where the dots are disks if $\omega < [\omega]$ and are stars if $\omega > [\omega]$.

As a technical resource, an elegant way to generate only at least silver medaling fractions, and hence our sequence q_n, for ω is to use the *optimal* continued fraction introduced by Bosma [1]. The following theorem is a refinement of Bosma's algorithm as given in [8, Chapter IX, Proposition 28], and is stated here without proof. The notation $tA \oplus B$, with $A = \frac{a}{b}$ and $B = \frac{c}{d}$, means

$$tA \oplus B = \frac{ta + c}{tb + d},$$

where a, b, c, d are integers, and t is a real number. The solution to $tA \oplus B = \omega$ is $t = \frac{c-\omega d}{\omega b - a}$. The symbols $C_{-2} = \frac{0}{1}$ and $C_{-1} = \frac{1}{0}$ are called *preconvergents*; when $n \geq 0$, C_n is called a *convergent*. Define $\text{sgn}(t)$ as the sign of t. Recall that $\lceil t \rceil$ is the *ceiling* of t.

Theorem 1: The optimal continued fraction for ω. For all integers $n \geq 0$, let $C_{n-1} = \frac{a}{b}$, $C_{n-2} = \frac{c}{d}$, $t = \frac{c-\omega d}{\omega b - a}$, $\epsilon = \text{sgn}(t)$, and

$$\delta = \frac{d + \epsilon b(\lceil |t| \rceil + 1)}{2d + \epsilon b(2\lceil |t| \rceil + 1)}.$$

Then with $m = \lceil |t| - \delta \rceil$, $C_n = \frac{p}{q}$ is at least a silver medaling approximation for ω where

$$C_n = mC_{n-1} \oplus \epsilon C_{n-2} = \frac{ma + \epsilon c}{mb + \epsilon d},$$

$m \geq 2$, $p = ma + \epsilon c > 0$, $q = mb + \epsilon d > 0$, $pb - qa = \pm 1$, and $\frac{p}{q}$ is in reduced form.

In Example 2, we use the Fibonacci numbers f_n, the first few terms of which are given in Table 1. Note that the third row of the table suggests that the sequence $\frac{f_{n+1}}{f_n}$ converges to $\phi \approx 1.61803$, as indeed it does.

Table 1 : *The Fibonacci numbers, $f_{j+2} = f_{j+1} + f_j$, $j \geq 0$.*

j	0	1	2	3	4	5	6	7	8	9	10	11	12
f_j	0	1	1	2	3	5	8	13	21	34	55	89	144
f_{j+1}/f_j	-	1	2	1.5	1.6667	1.6	1.625	1.6154	1.6190	1.6176	1.6182	1.6180	-

Example 2: Gold medaling approximations for ϕ via Theorem 1.

Step 0: The solution to $tC_{-1} \oplus C_{-2} = \frac{1 \cdot t + 0}{t \cdot 0 + 1} = \phi$ is $t = \phi$. So $\epsilon = 1$, $\delta = \frac{1+0}{2 \cdot 1 + 0} = \frac{1}{2}$, $m = \lceil \phi - \frac{1}{2} \rceil = 2$. Thus $C_0 = 2C_{-1} \oplus C_{-2} = \frac{2}{1}$, which is a gold medaling approximation for ϕ.

Step 1: The solution to $tC_0 \oplus C_{-1} = \phi$ is $t = \frac{1 - \phi \cdot 0}{\phi \cdot 1 - 2} \approx -2.618$, so $\epsilon = -1$. This time $\delta = \frac{0 + \epsilon \cdot 1 \cdot 4}{2 \cdot 0 + \epsilon \cdot 1 \cdot 7} = \frac{4}{7} \approx 0.57$, and $m = \lceil |t| - \frac{4}{7} \rceil = 3$, which means that $C_1 = 3C_0 \oplus \epsilon C_{-1} = \frac{5}{3} = \frac{f_5}{f_4}$. (As the reader may show, for all integers $k \geq 2$, $\frac{f_{k+1}}{f_k}$ is a gold medaling fraction for ϕ when k is even and a silver medaling fraction when k is odd.)

Step 2: The solution to $tC_1 \oplus C_0 = \phi$ is $t = \frac{2 - \phi \cdot 1}{\phi \cdot 3 - 5} \approx -2.618$. This time $\delta = \frac{11}{19} \approx 0.58$, which means that $C_2 = 3C_1 \oplus \epsilon C_0 = \frac{13}{8} = \frac{f_7}{f_6}$.

Step 3 and more: Again the solution to $tC_2 \oplus C_1 = \phi$ is $t = \frac{5 - 3\phi}{8\phi - 13} \approx -2.618$. This time $\delta = \frac{29}{60}$ which means that $C_3 = \frac{34}{21} = \frac{f_9}{f_8}$. Similarly, $C_4 = \frac{f_{11}}{f_{10}} = \frac{89}{55}$, $C_5 = \frac{233}{144}$, and in general $C_k = \frac{f_{2k+3}}{f_{2k+2}}$.

Example 3: The optimal continued fraction convergents for e and π. Applying Theorem 1 gives e's first few optimal convergents:

$$C_0 = \frac{3}{1}, C_1 = \frac{8}{3}, C_2 = \frac{19}{7}, C_3 = \frac{87}{32}, C_4 = \frac{193}{71}, C_5 = \frac{1264}{465},$$

and π's first few optimal convergents are

$$C_0 = \frac{3}{1}, C_1 = \frac{22}{7}, C_2 = \frac{355}{113}, C_3 = \frac{104348}{33215}. \tag{1}$$

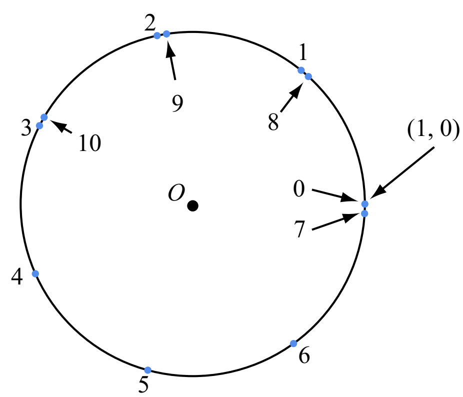
With respect to the sequences S_ω used to generate $\mathcal{F}_\omega$, from Examples 2 and 3 we have

$$S_\phi = \{1,\,3,\,8,\,21,\,55\}, \qquad S_e = \{1,\,3,\,7,\,32,\,71\}, \quad \text{and} \quad S_\pi = \{1,\,7,\,113\}.$$

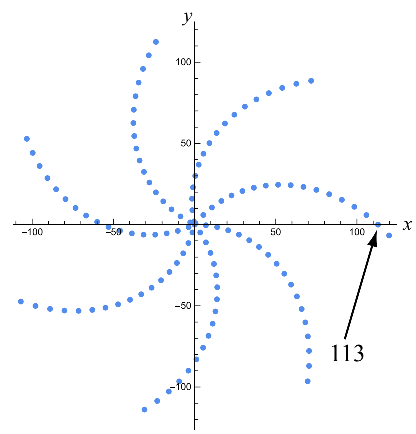
Thus Figure 1.a is $\mathcal{F}_\pi$, Figure 1.b is $\mathcal{F}_e$, and Figure 1.c is $\mathcal{F}_\phi$.

Why the flower analogy is so good

To see a deeper reason why these stylized flowers characterize continued fractions so well, imagine the unit circle centered at the origin O is rotating counterclockwise uniformly at ω revolutions per second. Refer to integer n as *index* n in seconds. Fix a point P on the circumference so that P at index n is at $P(n) = (\cos 2\pi\omega n,\, \sin 2\pi\omega n)$. At what indices n will P be near its initial position $P(0) = (1,\,0)$? Observe that $P(q) \approx P(0)$ if and only if $q\omega$ is near an integer number p of revolutions if and only if $p = [q\omega]$ if and only if $\frac{p}{q}$ is a good candidate for possibly being a medal-winning approximation for ω.



a. $P(n)$, $0 \le n \le 10$.



b. $\mathcal{F}_\pi$, $0 \le n \le 120$.

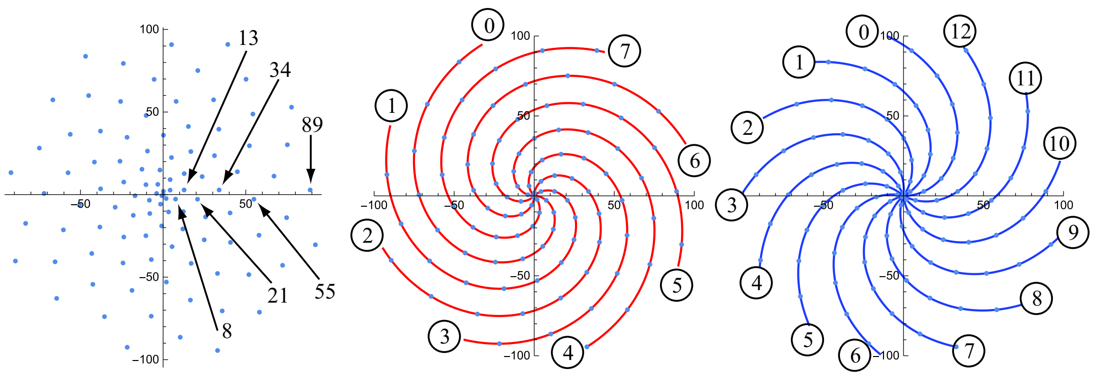
Figure 3 : *Points along the unit circle and an arithmetic flower.*

For example, let $\omega = \pi$. At any moment, P is $\pi - 3 \approx 0.14$ rotations counterclockwise from where it had been one second ago—approximately $2\pi \cdot (0.14) \approx 0.89$ radians $\approx 50.97°$. At index $q = 1$ second, then $p = 3$, and $P(1) \approx (0.63,\,0.78)$; that is, the fraction $\frac{3}{1}$ is trying to approximate π, but not very successfully. At index $q = 7$, then $p = 22$; this time $P(7) \approx (0.9985,\,-0.0556)$. And indeed, $\frac{22}{7}$ is a gold medalist for π. We could continue using this method of graphically finding indices q where $P(q) \approx (1,\,0)$, a flower model used in [4], but plotting $P(n)$ for all indices n soon leaves the unit circle cluttered with points atop of points, as is beginning to occur in Figure 3.a. However, at the cost of scaling points away from the positive x-axis (including the points that would otherwise have been near $(1,\,0)$), we could scale $P(n)$ away from O by a factor of n, resulting in what we call the *arithmetic flower* $\mathcal{F}_\omega$,

$$\mathcal{F}_\omega = \{n(\cos 2\pi\omega n,\, \sin 2\pi\omega n) |\, n \text{ is a nonnegative integer}\}.$$

Plotting points in an arithmetic flower soon leads to the discovery that its points seem to arrange themselves into a *family of spirals*. For example, Figure 3.b shows that the first 120 points of $\mathcal{F}_\pi$ appear to form a family of 7 spirals. Aha! Is it only a coincidence that 7 is the denominator of the optimal continued fraction convergent $C_1 = \frac{22}{7}$ from (1)? No; instead this pattern persists, and can be interpreted as a geometric continued fraction algorithm as shown in [8, Chapter 8]. For example, in Figure 3.b, locate the first spiral that appears to cross the positive x-axis as it emanates from O; this spiral appears to cross near the point with index $n = 113$, which means that the corresponding point on the unscaled unit circle should be very near

$(0,\ 1)$. Indeed, the next convergent from (1) is $C_2 = \frac{355}{113}$. Furthermore, if we plot about 10000 points of $\mathcal{F}_\pi$ we see that they have arranged themselves as a family of 113 spirals in Figure 5.b. The same phenomenon occurs with respect to the golden mean flower $\mathcal{F}_\phi$. In Figure 4.a, the points whose indices are Fibonacci numbers all lie near the positive x-axis—such points correspond to points on the unit circle near $(1,\ 0)$ of Figure 3.a. Furthermore, in Figure 4.b, spiral 1 (of eight spirals) is the first spiral to cross the positive x-axis, doing so near the points with indices $f_7 = 13$ and $f_8 = 21$; and from Example 2, $C_3 = \frac{34}{21}$. Similarly, in Figure 4.c, $\mathcal{F}_\phi$ appears as a family of $f_7 = 13$ spirals and spiral 12 is the first spiral to cross the positive x-axis, doing so near the points with indices 21 and 34, which again corresponds to C_3. The mathematical/biological literature abounds with articles describing the phenomenon of leaf and seed arrangements being that of a Fibonacci number of spirals. For example, see [4] and [5] which model the seed arrangement in the sunflower using a scaling factor of $\sqrt{n}$ so that $\sqrt{n}(\cos 2\pi\phi n,\ \sin 2\pi\phi n)$ represents the position of seed n in a sunflower—because as seeds form at the center O of the sunflower, they push the older seeds outward on the order of distance $\sqrt{n}$ from O. We use the scaling factor n in $\mathcal{F}_\omega$ rather than $\sqrt{n}$ so that index n and distance of $P(n)$ from O conveniently coincide for points in $\mathcal{F}_\omega$ near the positive x-axis.



a. $\mathcal{F}_\phi$, $0 \leq n \leq 100$. b. A family of 8 spirals. c. A family of 13 spirals.

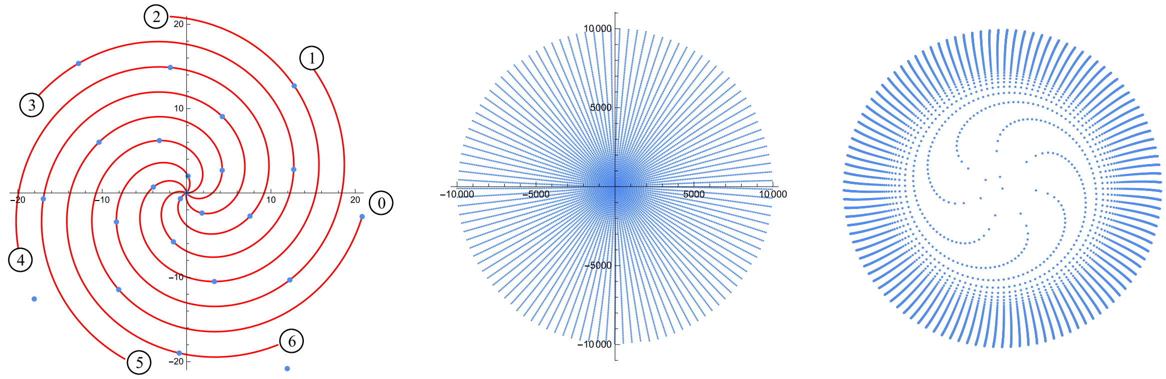
Figure 4 : *$\mathcal{F}_\phi$ as a collection of points and spiral families.*

Ideal spirals in the arithmetic flower

In this section we define what we mean by an apparent spiral within the arithmetic flower $\mathcal{F}_\omega$. As shown in [7] and [8] and as exemplified with the specific index $q = 8$ for $\mathcal{F}_\phi$ in Figure 4.b, we can imagine $\mathcal{F}_\omega$ to be a family of q continuous *ideal* spirals, denoted $L_r(t)$, where t is a nonnegative real parameter, r is an integer with $0 \leq r < q$, the period of each L_r is $T = \frac{1}{\omega - \frac{p}{q}}$, $p = [q\omega]$ with p and q being relatively prime. We allow periods to be negative numbers. In particular, $L_0(t)$ contains the points of $\mathcal{F}_\omega$ whose indices are nonnegative multiples of q. For each j, $1 \leq j < q$, there is a unique r, $1 \leq r < q$, where the graph of $L_r(t)$ contains the points of $\mathcal{F}_\omega$ whose indices are $qk + j$, where k is any nonnegative integer. With $p = [\omega q]$, p and q relatively prime, and $Q = \frac{1}{q\omega - p}$, namely, the period $T = \frac{1}{\omega - \frac{p}{q}}$ divided by q,

$$L_r(t) = t\left(\cos 2\pi(\omega - \frac{p}{q})(t + rQ),\ \sin 2\pi(\omega - \frac{p}{q})(t + rQ)\right). \tag{2}$$

For example, in Figure 4.b, $\omega = \phi$, $q = 8$, $p = 13$, $T = \frac{1}{\phi - \frac{13}{8}} \approx -143.55$, and $Q = \frac{T}{8} \approx -17.94$. In that figure, each spiral L_r is labeled with its subscript r. $L_0(t)$'s graph contains the points of $\mathcal{F}_\phi$ with indices, $0, 8, 16, 24$, and so on, while the graph of $L_1(t)$ contains points with indices, 5, 13, 21, 29 and so on; that is, when $r = 1$, then $j = 5$. The next theorem shows how j and r are related in general.



a. $\mathcal{F}_\phi$ as 7 spirals. b. $\mathcal{F}_\pi$, $0 \leq n \leq 10000$. c. $\mathcal{G}_\pi$, $0 \leq n \leq 10000$.

Figure 5 : *A gallery of abstract flowers.*

Theorem 4: A family of ideal spirals. Let ω be a positive irrational number and q be a positive integer relatively prime to $p = [q\omega]$. The family of curves $L_r(t)$, $0 \leq r < q$, from (2) are disjoint arithmetic spirals for $t > 0$ and every point on $\mathcal{F}_\omega$ belongs to $L_r(m)$ for some nonnegative integers m and r.

Proof. Let A and B be nonnegative numbers. The graph of parametric equations $f(t) = t(\cos A, \sin A)$ for $t \geq 0$ is a ray from the origin O through the point $(\cos A, \sin A)$. Let $g(t) = \begin{bmatrix} \cos Bt & -\sin Bt \\ \sin Bt & \cos Bt \end{bmatrix}$, a rotation matrix. Recall the identities $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$. Thus the graph of

$$g(t)f(t) = \begin{bmatrix} \cos Bt & -\sin Bt \\ \sin Bt & \cos Bt \end{bmatrix} \cdot f(t) = t\Big(\cos(tB+A),\, \sin(tB+A)\Big), \tag{3}$$

is an arithmetic spiral. Let $A = \frac{2\pi}{q}$. The family of curves $f_r(t) = t(\cos rA, \sin rA)$, $0 \leq r < q$, for $t > 0$ forms a wagon wheel of q radially symmetric straight-ray spokes about O. By (3), the family of the q curves $t(\cos(tB+rA), \sin(tB+rA))$ is a family of disjoint arithmetic spirals for $t > 0$ by continuity of the f_r's, because the q points from the straight-ray spokes at radial distance t from O are all distinct, and rotating them about the origin by tB preserves distinctness of points. With $A = 2\pi(\omega - \frac{p}{q})Q$ and $B = 2\pi(\omega - \frac{p}{q})$, then $L_r(t) = g(t)f_r(t)$ are disjoint arithmetic spirals for $t > 0$, where $0 \leq r < q$, $0 \leq j < q$, and $m = qk + j$ for all integers $k \geq 0$. Let $r \equiv pj \bmod q$. To finish the proof we show that

$$m(\cos 2\pi\omega m,\, \sin 2\pi\omega m) = m\left(\cos 2\pi(\omega - \frac{p}{q})(m + rQ),\, \sin 2\pi(\omega - \frac{p}{q})(m + rQ)\right).$$

For the first component of this identity we have

$$\begin{aligned}
\cos(2\pi(\omega - \tfrac{p}{q})(m + rQ)) &= \cos(2\pi(\omega - \tfrac{p}{q})(qk + j + rQ)) \\
&= \cos(2\pi\omega(qk+j) + 2\pi(-\tfrac{p}{q})(qk+j) + 2\pi\tfrac{(\omega - \frac{p}{q})r}{(\omega - \frac{p}{q})q}) \\
&= \cos(2\pi\omega m - 2\pi pk - \tfrac{2\pi pj}{q} + \tfrac{2\pi r}{q}) \\
&= \cos(2\pi\omega m + \tfrac{2\pi(r - pj)}{q}) = \cos(2\pi\omega m).
\end{aligned}$$

Since the similar result occurs when replacing cosine with sine, we are done. □

Because of the peculiar nature of the *jumbling* of the r's for the q spirals in the above proof, we consider the dynamics as shown in Table 2 for a particular q and ω, namely, $q = 8$ and $\omega = \phi$, so that $p = 13$. For

Table 2 : *The dynamics of r versus p and j, $r \equiv pj$ mod q with $q = 8$ and $p = 13$.*

j	0	1	2	3	4	5	6	7
$r \equiv pj \bmod q$	0	5	2	7	4	1	6	3

example, when $j = 1$, the spiral whose points from $\mathcal{F}_\phi$ have indices 1, 9, 17, ..., corresponds with spiral $L_5(t)$, and so on.

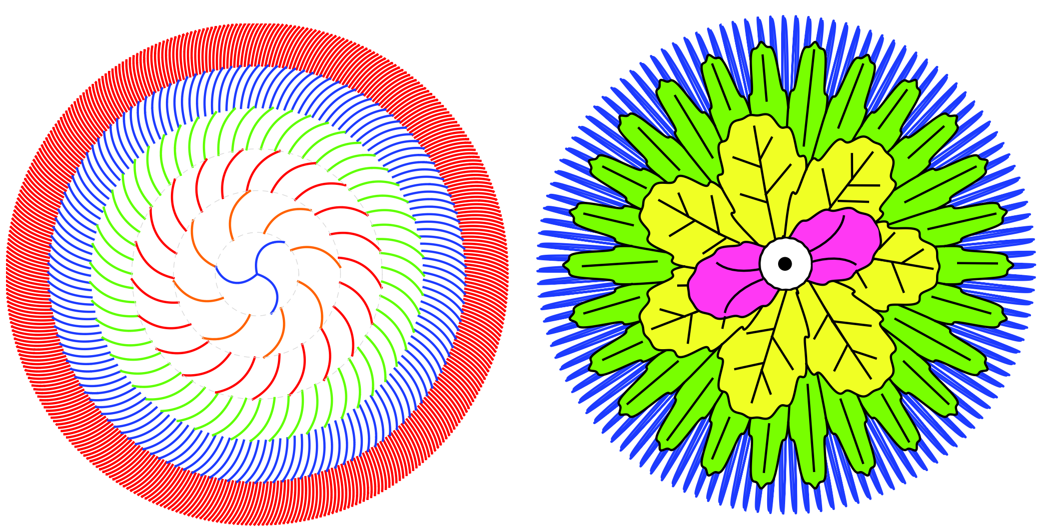
As an example illustrating a family of spirals for $\mathcal{F}_\phi$ when $\frac{p}{q}$ fails to be a medal-winning fraction for ϕ, let $q = 7$ whereupon $p = [7\phi] = 11$ and $|\phi - \frac{11}{7}| \approx 0.047 > 0.0102 \approx \frac{1}{2 \cdot 7^2}$. As shown in Figure 5.a, the period of each ideal spiral is about 21.46, which means that any single rotation of ideal spiral 0 about O contains only 3 or 4 points of $\mathcal{F}_\phi$ whereas when $\frac{p}{q} = \frac{13}{8}$, as in Figure 4.b, each single rotation of a spiral about O contains about 18 points.

Whenever $\frac{p}{q}$ is a medal-winning fraction for ω, then $|\omega - \frac{p}{q}|$ is near 0. Thus the period $T = \frac{1}{\omega - \frac{p}{q}}$ of each of the q ideal spirals will be a fairly large number. This large number divided by q gives the relatively large (compared to q) approximate number of $\mathcal{F}_\omega$'s points on any single rotation of any spiral—which is why the human eye can perceive the apparent families of spirals in, say, Figure 4.a, for various integers q corresponding to medal-winning fractions $\frac{p}{q}$ to ω.

To focus on another apparent aspect, note that the appearance of $\mathcal{F}_\omega$ changes qualitatively as index n increases. For example, $\mathcal{F}_\pi$ appears as a family of 7 spirals, $0 \le n \le 100$, as shown in Figure 3.b. However as n increases, the once pronounced appearance of 7 spirals fades, and is replaced with, in the case of Figure 5.b where $0 \le n \le 10000$, the appearance of a family of 113 spirals. In order to view this progression of different families of spirals within a single flower, we proceed as in [6] and warp space by replacing the scaling factor n in $\mathcal{F}_\omega$ with $\ln n$. Thus we say that the *logarithmic spiral*, denoted $\mathcal{G}_\omega$, is the set of points

$$\mathcal{G}_\omega = \{\ln n(\cos 2\pi\omega n, \sin 2\pi\omega n) |\, n \text{ is a positive integer}\}.$$

In Figure 5.c, $\mathcal{G}_\pi$ clearly displays the transition from a family of 7 spirals to that of 113 spirals. Figure 6.a shows successive annuli of families of spirals as index n increases for $\mathcal{G}_\phi$—a collage formed by clipping the various families of spirals in their indices of dominance. Replacing each family of these short arcs with a corona layer of petals results in a more artfully rendered flower of Figure 1.c, namely $\mathcal{F}_\phi$.



a. Logarithmic spiral layers for $\mathcal{G}_\phi$. b. An unknown flower: $\mathcal{F}_x$.

Figure 6 : *A logarithmic skeleton and an unknown real number x.*

Reverse engineering a flower

Given a flower $\mathcal{F}_x$ as in Figure 6.b, where x is unknown, we wish to find a good approximation to x. Counting the petals in the successive layers of the flower, we can find S_x, the sequence of petal numbers. Can you guess the medal-winning fractions that are implicitly evident in the flower of Figure 6.b?

Step 0: We see that $S_x = \{1, 2, 7, 26, 123\}$. Since the center of the flower has but a single circular dot, we know that $\frac{1}{2} < x < 1$. So $C_0 = 1$.

Step 1: We know that $C_1 = \frac{w}{2}$ for some integer w, $0 \leq w \leq 1$, and that w and 2 are relatively prime. So $w = 1$ and $C_1 = \frac{1}{2}$.

Step 2: We know that $C_2 = \frac{|z|}{7}$ and that $7y + 2z = 1$ for some integers y and z. By Euclid's algorithm for determining the greatest common divisor of two positive integers, one solution to this Diophantine equation is $y_0 = 1$ and $z_0 = -3$, and thus from elementary number theory (see [2, pp. 11–12]) any solution to the Diophantine equation is $y = y_0 + 2k = 1 + 2k$ and $z = z_0 - 7k = -3 - 7k$ for any integer k. When $k = 0$, then $\frac{|y|}{2} = \frac{1}{2}$ and $\frac{|z|}{7} = \frac{3}{7}$. The only other possible solution is when $k = -1$, which gives $\frac{|y|}{2} = \frac{1}{2}$ and $\frac{|z|}{7} = \frac{4}{7}$. So C_2 is either $\frac{3}{7}$ or $\frac{4}{7}$. Since $|\frac{3}{7} - \frac{1}{2}| = |\frac{4}{7} - \frac{1}{2}|$ and $x > \frac{1}{2}$, then $C_2 = \frac{4}{7} \approx 0.5714$.

Step 3: We know that $C_3 = \frac{|s|}{26}$ and that $26r + 7s = 1$ for some integers r and s. By Euclid's algorithm we know that one solution is $r_0 = 3$ and $s_0 = -11$, so $r = 3 + 7k$ and $s = -11 - 26k$ is a solution to the Diophantine equation for all integers k. We know that $|r| = 4$ which corresponds with $k = -1$, which means that $s = 15$, and so $C_3 = \frac{15}{26} \approx 0.576923$.

Step 4: Similarly, we can reason that $C_4 = \frac{71}{123} \approx 0.577236$. You might now recognize that these convergents are near Euler's gamma $\gamma \approx 0.577216$, and thus we might view Figure 6.b as $\mathcal{F}_\gamma$.

Using the above approach we can similarly reverse engineer $\mathcal{F}_\omega$ for any ω.

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