A Hands-on Laboratory with Mathematical Mechanical Drawing Machines

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Abstract

We describe a hands-on activity involving mathematical mechanical drawing machines. In particular, attending participants will construct a parabolograph, an artifact which traces rigorously a parabola with a *continuous* line. The workshop is addressed to general public and young people from 13-14 years old. The ultimate goal is to lead the participants to make a direct connection between the shape of the plotted curve and the setting of the machine, which is closely associated with the definition of the curve as a geometric locus. The approach reflects the historical cultural content of these machines, dating from XVII century. This approach has been tested by the authors in laboratorial sessions with high school students, in an elective course of the school of Architecture of Roma Tre University and in occasion of the European Researchers' Night 2017.

Introduction: the Mathematical Drawing Machines

This paper shows a hands-on activity based on the construction of a mechanical mathematical machine, a parabolograph.

> *A mathematical machine has as its fundamental purpose to constrain a point, or a segment, or any figure to move in space or undergo transformations exactly following an abstract mathematical law.* [2].

The workshop is addressed to general public and young people from 13-14 years old, and could be suitable for high school classes. The main goal is to lead participants to understand the connection between the shape of the plotted curve and its definition as a geometric locus; this connection passes through the manipulation of the machine and its settings.

The authors developed an activity for high schools, over the same topics[1] and have been conducting for some years an elective course in the School of Architecture of Roma Tre University [8]. The course Mathematical Drawing Machines combines the hands-on approach with the analytical geometrical treatment and the study of some cases found in the treatises of history of architecture ([6,7], Figure 5). Moreover hands-on approach is very much advocated also in other math courses and activities in the School of Architecture of Roma Tre University [10], in the path of a dynamic teaching of mathematics [5]. We learned from Reza Sarhangi how a well-planned workshop could be effective also for a quite high number of students, and thus we decided to put into a workshop the material we had studied with historical methods [7]

When putting hands on a drawing machine, users test how the initial "setting" influences the shape of the curve. This is the seed of the "parametric thought". At the same time they visualize the curve and the corresponding analytical representation. Mathematical mechanical machines can be of many different kind (see for example [1], [2] [9]); in this workshop we propose a *thread machine*. The choice is not

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fortuitous, since thread conicographs embody the geometric locus corresponding to the curve, in a more direct way compared with other machines, and therefore constitute the ideal tool for trasmitting its comprehension. All the three conic sections: ellipses, parabola and hyperbola can be drawn with a thread mechanism [2,6]. The choice to propose a parabolograph for the Bridges audience was also motivated by the fact that during the 2017 European Researchers' Night, we recorded enthusiastic reactions in front of this machine, from teachers, families, teenagers. This activity is well suited for a group of 20-25. Youngsters are today used to numerical simulations, which are necessarily drawn over a discretization. This workshop acquaints them with a mechanical treatment, which is closer to a continuous correct imagination. Infact these machines were born to draw curves with a continuos line, for various purposes, for example to cut hyperbolic lenses:

> *In the mathematical works of Descartes the theoretical machines prevailed, destined to a conceptual use, inserted in the theoretical discourse with completely independent function from a possible physical existence: it is evident that the author has never thought about their concrete realization, which would result however, in many cases, difficult or impossible. In the X Discourse of the "Diottrica" (and in a correspondence of 1629 with Ferrier, famous Parisian technician, maker of mathematical instruments) Descartes instead describes a real machine tool, conceived and designed for a practical use: the construction of hyperbolic lenses (which, would have avoided spherical aberrations when used in optical devices).*[11]

Structure of the Workshop

First Part: Introduction to Conics and to Parabola

The workshop foresees a first part of explanation of the underlying mathematical concepts, and also the illustration of the activity itself. We will introduce conic curves as planar sections of a conic surface, by showing some short digital animations. We will then move on to the metric definition of a parabola and its Cartesian equation. This introductory part aims to emphasize the fact that the shape of a parabola is given as soon as focus and directrix are given, and that the only parameter that influences its shape is the distance between these two geometrical entities. We will show, by classic drawing on a blackboard, some of the focal properties of the parabola. We will also show to the participants the materials they will use, as well as a ready model of the machine.

Participants will work in teams of two. This is necessary and advisable: the two participants will have to communicate with each other and make decisions (at what distance to place the focus from the directrix?); this is part of the learning process. Moreover, they will need four hands, not two, to activate the “machine” (Figure 4). In fact, it is possible to construct machines that work even if only one person operates them, but they would require more sophisticated and costly materials, such as wood. It is one aspect illustrating when working in teams is more effective.

The parabola is *the geometric locus of the points of the plane equidistant from a point called focus and from a straight line called directrix*. Starting by this sentence we will show some of the properties of the parabola, by drawing on a blackboard, then move forward to the second part.

Second Part : Assembling the Machine

The materials which we will use to assemble the machine are displayed in figure 1: a sheet of *poliplat*, i.e. a polystyrene slab coupled between two sheets of very light cardboard on the sides. Very light, it will be easy to place nails in it, and one can draw on it. Then participants will use nails, colored thread, tape (Figure 1, left).

On the 2-dimensional support fix a poliplat ruler (Figure 1, right) which plays the role of the directrix *d*, choose a point (focus *F*) and fix a nail on *F*. Place the poliplat set square perpendicular to the directrix, and fix another nail on point *A* on the set square (Figure 2).

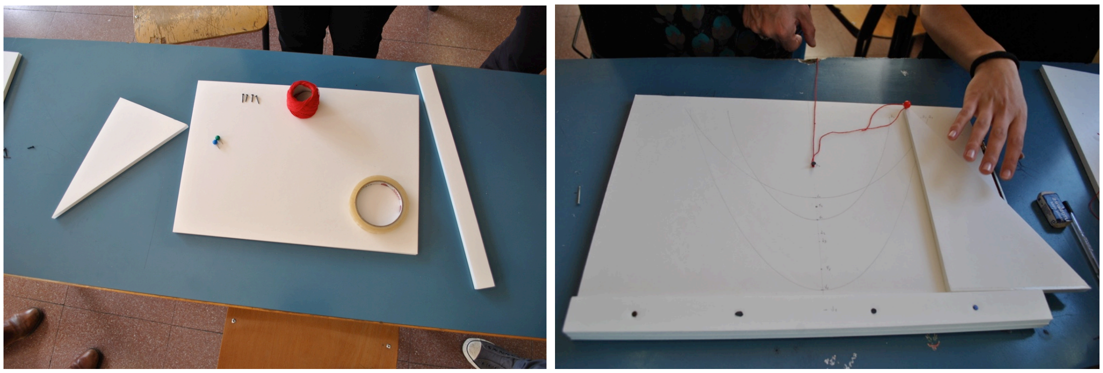

Figure 1: *Left: the materials to construct a thread parabolograph. Right: the assembled machine*

The length of the thread must be equal to the distance between *A* and the directrix (this is crucial to obtain a parabola). The thread is measured and then fixed at points *F* and *A*. The square set will be shifted along the directrix; it is advisable to put some tape on the side of the square set which is in contact with the directrix, in order to facilitate the movement. Place the pencil in *A*, keeping it attached to the set square. By sliding the set square on the directrix the pencil draws the first half of the parabola. At this point it will be necessary to pull out the nail in *A*, turn the set square, reposition the nail on the opposite side of the triangle (Figure 3) and draw the second half of the curve.

This phase will take 40-50 minutes. All the participants work, in couples, to their machines, while the leader moves among them to help solve issues.
Mathematics: when everybody has drawn a curve, it is time to prove rigorously that the curve they traced is really a parabola. In order to do it, imagine to place a Cartesian reference with the axis of the abscissa coinciding with the directrix, and the axis of the ordinates passing through the focus (Figure 3). The distance between the focus *F* and the directrix *d* will be indicated with letter *a*.

Figure 2 right image: (*A'P'+P'F*) is equal to the length of the thread.
(*A'P'+P'K'*) is also equal to *A'K*' which is the length of the set square. Since we fixed nails in the way to have AK equal to the length of the thread, we can conclude that *P'F=P'K'*, which means that the point *P* belongs to the parabola since it verifies the metric definition of the curve.

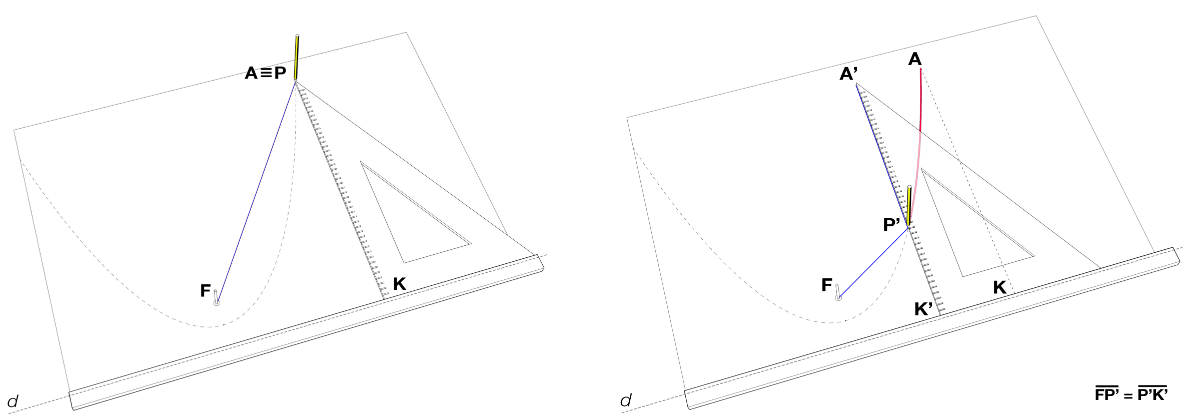

Figure 2: *How the parabolograph works [6]. Figure by Enrico Mele, architect.*

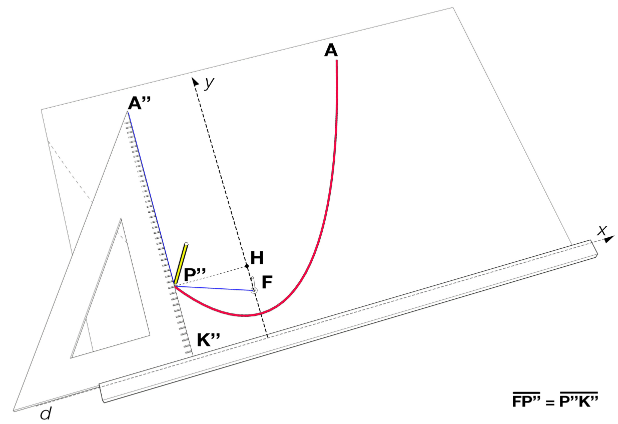

Figure 3: *How to deduce the cartesian equation of the plotted curve [6]. Figure by Enrico Mele, architect.*

In order to write down the cartesian equation of the plotted curve, we will recall that the definition of parabola yields that the distance *P"F* (Figure. 3) must be equal to the distance between *P"* and *d*, that is *P"K"*. Let (*x,y*) be the coordinates of the point *P"*. Then *P"H=x*, *P"K"=y*. Futhermore consider the right triangle *P"HF*, by Pithagoras' theorem:

$$FP'' = \sqrt{x^2 + (y-a)^2}$$

the equation *FP"= P"K"* yields

$$y = \sqrt{x^2 + (y-a)^2}$$

squaring both members and doing straightforward computations leads to

$$y = \frac{x^2}{2a} + \frac{a}{2}$$

which is the familiar equation of a parabola. The parameter a , the focus-directrix distance is responsible for the shape of the curve: moving away the focus from the directrix, allows to obtain a wider parabola.

Third Part: Let Them Experiment

We ask the participants to change the settings of their machine, one at a time (the length of the thread, the position of the focus, the position of the point *A*) and draw another parabola. They can proceed by arbitrarily changing the setting, and then look at the new curve, or first plan how they want to change the curve, and then act on the mechanism. The aim is to understand how the change in the settings affects the shape of the curve. Since the only parameter which affects the form of the parabola is the distance focus-directrix, we want them to associate these two ideas.

Figure 4: *The authors while using a wooden prototype of the machine.*

Moreover, in our experience, after assembling the artifact the participants are willing to try new configurations of the machine, sometimes it is unnecessary to ask them to change something: they spontaneously do it. In any case there will be very little time left to dedicate to this phase of free experimentation, since the assembling and the tracing will occupy almost all the available time.

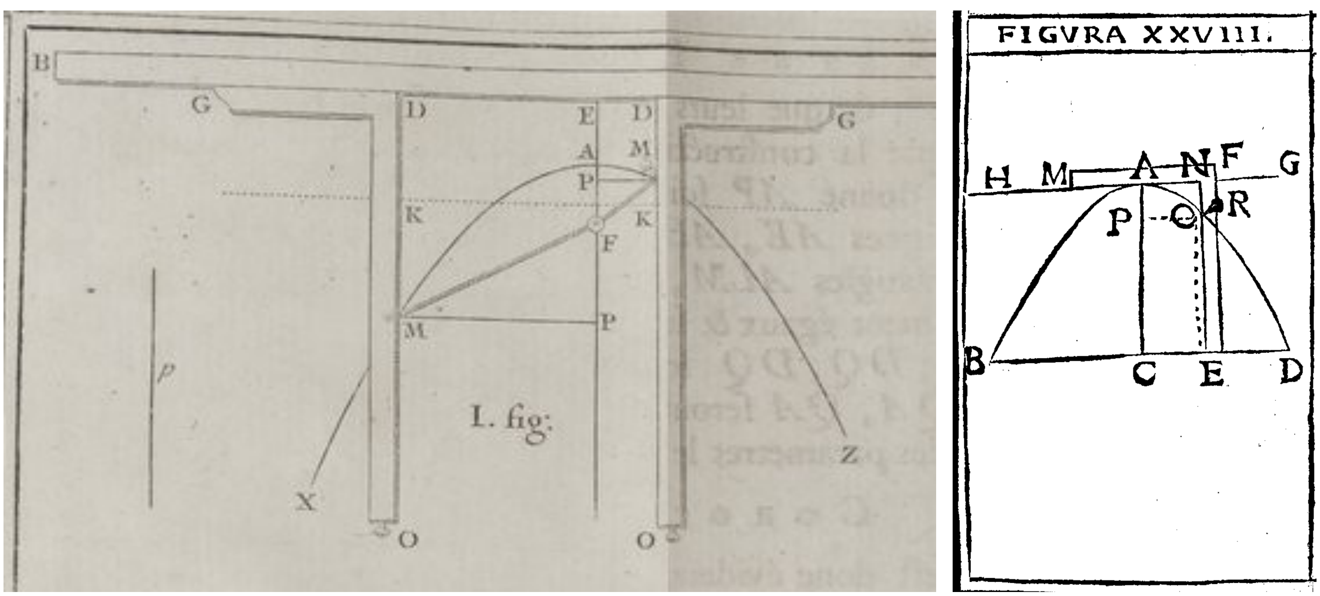

Figure 5: *The thread parabolograph in two historical treatises: left [4]; right [3]*

Conclusions

The hands-on workshop described in this paper can be realized during school hours and is suitable for high school classes. The manual activity lasts about 45 minutes, so, if the students have already been explained the mathematical skills, the construction of the machine can be included in a one hour lesson.

The polistirene can be replaced by cardboard at least 5mm thick, and thread machines to draw ellipses and hyperbola can be proposed [6] as well. The Italian Mathematical Union (UMI-CIIM), includes drawing machines as valuable learning tools, as well as dynamical geometry software in the document which describes the curriculum for secondary schools mathematics.

This workshop is an example of dynamical teaching of mathematics [5], aiming to connect the analytic representation of curves, to their drawing, and their shape. We would like mathematical concepts to pass also through the senses, like sight and touch, offering an alternative to the traditional frontal lesson, and aiming to activate students.

Acknowledgements

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References

[1] E. Barbin. *Les Constructions Mathématiques Avec des Instruments et des Gestes.* Ellipses Paris. Collection : IREM - Epistémologie et Histoire des Mathématiques. 2014.

[2] M.G.C. Bartolini Bussi and M. Maschietto. *Macchine Matematiche: dalla Storia alla Scuola*. Springer. 2006.

[3] B. Cavalieri. *Lo Specchio Ustorio Overo Trattato delle Settioni Coniche et Alcuni Loro Mirabili Effetti Intorno al Lume, Caldo, Freddo, Suono e Moto Ancora*. Ed. Clemente Ferroni. 1632. Source gallica.bnf.fr / Bibliothèque nationale de France.

[4] G. F.A. de L'Hospital. *Traité Analytique des Sections Coniques et de Leur Usage Pour la Résolution Des Équations*. Moutard (Paris). 1776.

[5] F. Enriques. "L'insegnamento Dinamico." *Periodico di Matematiche* s. IV. vol. 1. 1921. pp 6-16.

[6] L. Farroni and P. Magrone. "Mathematical Drawing Machines: Historic Drawing From a Parametric Point of View. The Case of Conic Curves.*" Proceedings Congresso di* EGRAFIA. 1-3 octubre 2014. Rosario. CUES. vol. 1. pp. 130-137. (ISBN: 9788897821809).

[7] L. Farroni and P. Magrone. "A Multidisciplinary Approach to Teaching Mathematics and Architectural Representation: Historical Drawing Machines. Relations Between Mathematics and Drawing." *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics*. L. Radford, F. Furinghetti, T. Hausberger eds. Montpellier, IREM de Montpellier. pp. 641-651.

[8] Drawing Machines Elective Course in Roma Tre. http://www.formulas.it/?page_id=5299.

[9] Mathematical Machines Association. http://www.macchinematematiche.org/.

[10] Formulas Workshops. http://www.formulas.it/?page_id=5974.

[11] Teatrum Machinarum Exhibition. Description of the Machine by Descartes http://archivioweb.unimore.it/theatrum/macchine/008ogg.htm.