3D Printing Chaos

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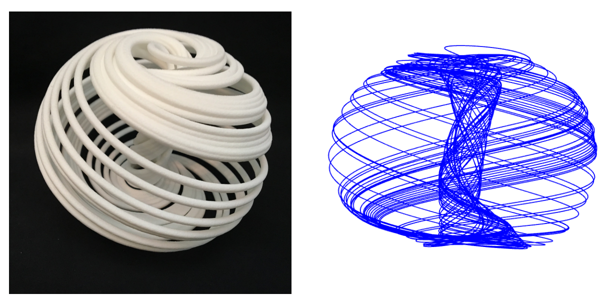
Abstract

In this paper, we use 3D prints to demonstrate long-term behavior of solutions to the Aizawa equations for various parameters, some of which admit chaos. The methods for creating the prints produce high quality meshes and can be used with other nonlinear dynamical systems.

Introduction

Chaos is described as a sensitive dependence on initial conditions. The double-pendulum is a classic example of chaos, where two nearby starting positions give drastically different paths of the pendulum in the long term. In this paper, we use 3D printed solutions to visualize the long-term behavior of the Aizawa equations, a nonlinear dynamical system for which certain parameters result in chaotic solutions. Previous work on visualizing chaotic solutions in general [1] and the Aizawa equations specifically [3] do not utilize 3D printing. While solutions to other dynamical systems have been printed [6], we are not aware of 3D printing being used to investigate long-term behavior of chaotic solutions to the Aizawa equations.

An advantage 3D printing has over traditional computer-mediated visualizations is that 3D prints exist in the same dimension as the solutions. Furthermore, depth need not be interpreted like it does on a computer screen because in a 3D print depth can be seen and felt. For instance, compare the shadows and depth of Figure 1(a) with the computer generated Figure 1(b). Of course, Figure 1(a) is still a 2D representation of the actual 3D print, but even so, it provides a clearer picture of the solution.



(a) *3D Print* (b) *Software Visualization*

Figure 1: *Aizawa Attractor* $\alpha = 0.95$

In what follows, we first discuss basic information about nonlinear dynamical systems and chaos. Next, we discuss the Aizawa equation and a result that indicates which parameters may lead to interesting long-term behavior. We then show the prints and explain how they demonstrate the long-term behavior of the solutions. Finally, we discuss the printing technique used and apply it to other dynamical systems.

Nonlinear Dynamical Systems and Chaos

A nonlinear dynamical system on $\mathbb{R}^n$ is a system of ordinary differential equations of the form

$$\dot{\mathbf{x}} = F(\mathbf{x})$$

where $\mathbf{x} = (x_1, \cdots, x_n)$, $\dot{\mathbf{x}} = (\frac{dx_1}{dt}, \cdots \frac{dx_n}{dt})$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function which in this setting is also called a vector field. A solution to these equations is a function $\phi : \mathbb{R} \to \mathbb{R}^n$ such that $\dot{\phi}(t) = F(\phi(t))$ for all $t \in \mathbb{R}$. It is also useful to think of a solution as a flow that depends on an initial condition $\mathbf{x_0} \in \mathbb{R}^n$.

For instance, if the vector field represents the velocity of wind in a region then the flow would be the path a balloon would take in the wind.

To discuss whether a solution is chaotic or not, we need a definition of chaos. However, there is no singular agreed-upon definition of chaos and most definitions require a background in nonlinear dynamics. We give an intuitive description of chaos based on the definition given by Devaney, Hirsh, and Smale [4]. Intuitively, chaos occurs when:

1. solutions exhibit a sensitivity on initial conditions. (i.e. solutions starting near each other do not necessarily exhibit similar behavior); and
2. nearby sets of initial conditions will eventually spread out under the flow.

Proving a solution satisfies a definition of chaos is difficult. We can give evidence using numerical methods to estimate Lyapunov exponents [5], which describe the long-term 'spread' of nearby points in phase space. We can use numerical techniques [7] to compute Lyapunov exponents and a classification scheme [2] to justify our claims about the long-term behavior of solutions.

Aizawa Equations

The Aizawa equations given in (1) admit chaotic solutions for certain parameter/initial value combinations. However, not all of the solutions are chaotic even when they are visually similar to chaotic solutions. The Aizawa equations are

$$\begin{aligned}\dot{x} &= (z-\beta)x - \delta y \\ \dot{y} &= (z-\beta)y + \delta x \\ \dot{z} &= \gamma + \alpha z - \frac{z^3}{3} - \left(x^2+y^2\right)(1+\epsilon z) + \zeta z x^3\end{aligned} \tag{1}$$

for $(x, y, z) \in \mathbb{R}^3$ and parameters $\alpha, \beta, \delta, \epsilon, \gamma$, and ζ. Commonly used values of the parameters in visualizations are $\alpha = 0.95, \beta = 0.7, \delta = 3.5, \epsilon = 0.25, \gamma = 0.6$, and $\xi = 0.1$. The solution to the Aizawa equation with these parameters and initial value $\mathbf{x}_0 = (0.1, 0, 0)$ is pictured in Figure 1.

Following the analysis by Cope [3], we see that the value of α affects the number of equilibrium points and hence the dynamics of the system. We omit the proof, but we can show that

Theorem 1. *If $\alpha \geq \sqrt[3]{\frac{9\gamma^2}{4}}$ then the differential equations will have three real equilibrium points (counting multiplicity) and if $\alpha < \sqrt[3]{\frac{9\gamma^2}{4}}$ the differential equation will have one real equilibrium point.*

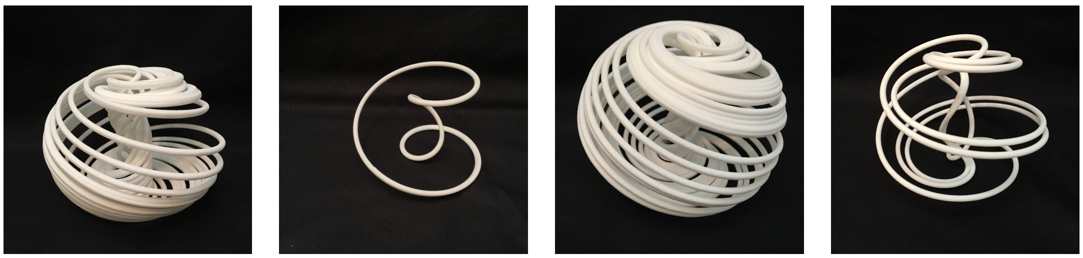
In the examples that follow, we will see three types of long-term behavior, chaotic, periodic, and quasi-periodic. Periodic solutions will repeat and we should expect to see a loop in the long-term behavior. Quasi-periodic solutions 'almost' repeat and we will still see a loop in the long-term behavior. Chaotic solutions will not settle down and the long-term behavior will not form a loop.

Since the number and location of the equilibrium points depend on α, we will vary α while keeping all other parameters constant. As we increase α from 0.93 to 1.06517 the equilibrium points will move further away from each other along the z-axis which causes the 'central tube' to become thinner until it disappears completely and the solutions drastically change shape. To demonstrate the long-term behavior, prints are shown for both $0 \leq t \leq 120$ and $100 \leq t \leq 120$. While long-term refers to the behavior as $t \to \infty$, we found that $t = 120$ is sufficient for these solutions to match behavior predicted by the Lyapunov exponents. Finally, we use $(0.1, 0, 0)$ as the initial value for each solution.

The solution for $\alpha = 0.93$ (which is slightly smaller than the bifurcation point $\alpha \approx 0.932$, Theorem 1) can be seen in Figure 2(a). While the solution may seem 'chaotic', we can see from Figure 2(b) it is in fact asymptotic to a periodic attractor.

In contrast, the solution for $\alpha = 0.95$ in Figure 2(c) initially appears very similar to the previous solution. However, with the introduction of two new equilibrium points, the long-term behavior (Figure 2(d)) does not appear to settle down which is characteristic of a chaotic solution.

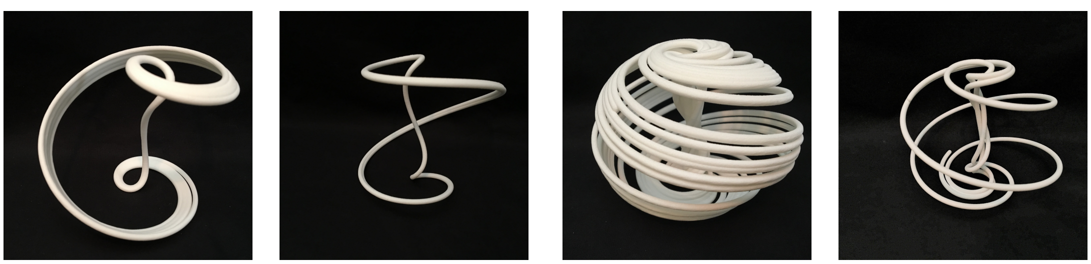
The solutions for $\alpha = 0.97$ and $\alpha = 0.99$ (Figure 3) have drastically different initial and long-term behaviors. When $\alpha = 0.97$, the solution quickly becomes close to the quasi-periodic attractor, whereas for $\alpha = 0.99$ the solution is chaotic.



(a) $0 \leq t \leq 120$ (b) $100 \leq t \leq 120$ (c) $0 \leq t \leq 120$ (d) $100 \leq t \leq 120$

Figure 2: $\alpha = 0.93$ *(a and b)* $\alpha = 0.95$ *(c and d)*

Another interesting change in the behavior of solutions occurs near $\alpha = 1.061$ where the solutions go from chaotic to convergent. The solution shown in Figure 4(a) for $\alpha = 1.05$ is asymptotic to the chaotic attractor. However, a small change to $\alpha = 1.061$ yields the solution in Figure 4(b) which starts similarly to the other chaotic solutions but then actually converges to an equilibrium point. Finally, with $\alpha = 1.0617$, we see the solution in Figure 4(c) rapidly converge to the same equilibrium point.



(a) $0 \leq t \leq 120$ (b) $100 \leq t \leq 120$ (c) $0 \leq t \leq 120$ (d) $100 \leq t \leq 120$

Figure 3: $\alpha = 0.97$ *(a and b)* $\alpha = 0.99$ *(c and d)*

(a) $\alpha = 1.05$ (b) $\alpha = 1.061$ (c) $\alpha = 1.0617$

Figure 4: *Various* α, $0 \leq t \leq 120$

The Printing Technique

To create the 3D prints, we use Mathematica software to numerically solve the differential equation and a Python program to import the solution into the Rhino software where we convert the data into a high quality mesh of cylindrical tubing. A similar method [6] uses a numerical solver included with Mathematica and then creates a tube around the solution. We found that for some differential equations, this solver produces jagged solutions even with very small step sizes. In addition, the tube command in Mathematica creates meshes with hexagonal tubes instead of cylindrical tubes which we did not find aesthetically desirable. To overcome these two issues, we use Euler's method with a very small step size in Mathematica to create a sequence of points that are then exported into a .csv file for further processing in Rhino.

We use Python code to both import the points into Rhino and interpolate a curve through the points. Next, we use Rhino to create a cylindrical pipe around the solution. Finally, we create a very fine mesh ($\sim$10 million faces) and reduce the mesh to meet the technical specifications required by Shapeways. By using Shapeways, we avoid the problem of the mesh having a high number of self-intersections. In the end, we have a very smooth print of the solution.

This technique allows us to make prints for any differential equation in 2 or 3 dimensions. In addition to being aesthetically pleasing, these prints can be used to aid with classroom instruction. To demonstrate the robustness of the technique, we leave the reader with three more prints of chaotic solutions in Figure 5.

(a) *Dequan Li* (b) *Lorenz* (c) *Anishchenko-Astakhov*

Figure 5: *Other attractors*

References

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