Star Origami

Joy Hsiao
Dept. of Mathematics,
Stuyvesant High School
345 Chambers Street, New York, NY 10282, USA
jhsiao@schools.nyc.gov

Abstract

A modular pentagonal star, whose creator is unknown, and a modular decagonal star designed by Tomoko Fuse will be folded using five and ten pieces of square paper, respectively. Angle measures of the decagonal star will be calculated to show how well the modular pieces fit together. Does each piece contribute exactly 36° to form a 360° circle? Participants will calculate angle measures in two different ways: by hand using high school mathematics, and on GeoGebra, a web-based graphing tool.

Introduction

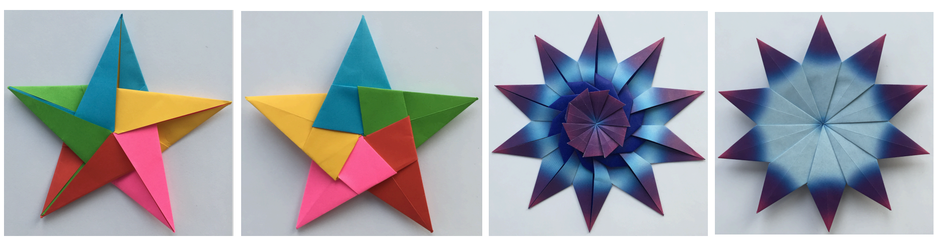

Figure 1: *Pentagonal and decagonal stars*

Star origami models are popular, especially around some holidays. They are often admired for their symmetries and repetitive patterns. Some star origami models are folded with one square piece of paper to form a four-point or eight-point star. Pentagonal paper may be folded to form a five-point star, and hexagonal paper may be folded to form a six-point star. Modular origami stars are made by folding multiple sheets of square paper and then assembled to form a multi-pointed star. It is therefore less obvious as for why square pieces of paper can be folded to form angle measures that meet perfectly to form a five-point or a ten-point star.

In this workshop, we will begin by folding both models of star origami. I learned how to fold the five-point modular star origami from a former student who learned it from her grandmother. At first, I was annoyed by how the pieces did not fit perfectly and questioned my folding skills. I then set out to prove that the design does not produce a perfect 72° angle as required in a five-point star. As a result of this study, I noticed how the star pops slightly into 3D when assembled. The five-point star model was analyzed in my 2015 article in Mathematics Teacher [1]. I will focus on the ten-point star in this workshop.

The decagonal star designed by Tomoko Fuse is made from folding and assembling ten pieces of square paper. Unlike the five-point star, I was intrigued by how these modular pieces appear to fit together perfectly, and again, I set out to find out if the measure of the angle from each modular piece is exactly 36° to better understand this model. During the workshop, we will calculate the angle measure by hand and with GeoGebra, an online graphing tool. Participants can follow along on their own computer or sit back and enjoy a demonstration.

As an educator, I highly recommend origami explorations where there are numerous opportunities for students to discover patterns, pose original questions, and apply their mathematical knowledge to find original solutions. In the process of finding angle or line segment measurements, all three branches of high school mathematics, algebra, geometry, and trigonometry, are almost always applied. In addition, in order to construct the crease pattern using a graphing tool, knowledge of the use of compass and straight-edge constructions and geometric properties are required. Once constructed, measurements can also be calculated on the graphing tool to confirm calculations done by hand. These exercises create real-life applications for geometric constructions and the use of mathematics in general. They provide many opportunities for students to be creative in finding their own solutions through various possible methods – pure mathematical calculations on paper, use of a graphing tool, or the actual folding of paper that often makes symmetries and congruence obvious. It is a fun way to put one's mathematical knowledge to practice.

Diagrams

The diagrams below are shown from a mathematical perspective where bisected line segments and angles are indicated and each bisector is a crease line. The folding instructions only lead up to the establishment of the angle used to assemble the decagonal star – the angle that helps us determine the "fit" of this model. Complete instructions on folding the decagonal star can be found in Fuse's book [2] or at this Bridges workshop.

In the diagrams below, solid line segments represent the edges of the square paper and dotted line segments represent a crease, a hidden crease, or an auxiliary line.

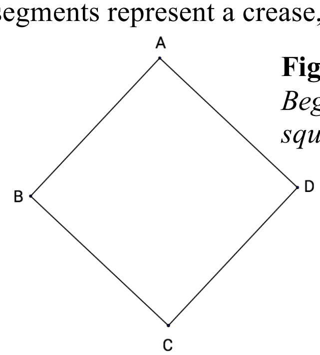

Figure 2: *Begin with a square paper*

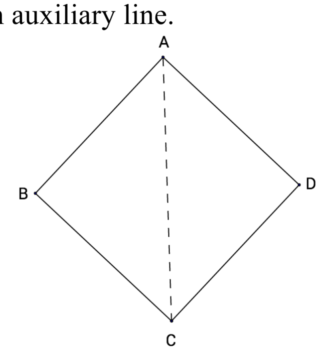

Figure 3: *Bisect $\angle BAD$* by folding D onto B, then unfold

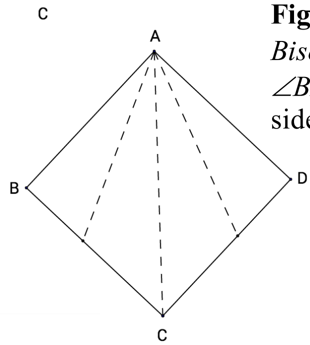

Figure 4: *Bisect $\angle DAC$ and $\angle BAC$*, keep the sides folded

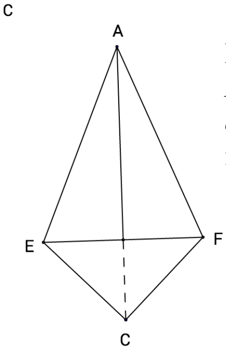

Figure 5: *Reflect point C over $\overline{EF}$*, keep it folded

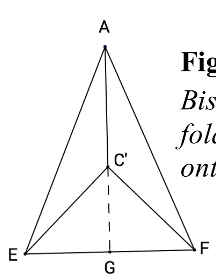

Figure 6: *Bisect $\overline{C'G}$ by folding C' onto G*

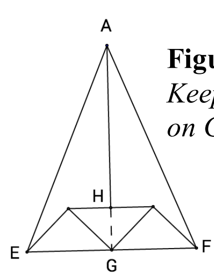

Figure 7: *Keep C' folded on G*

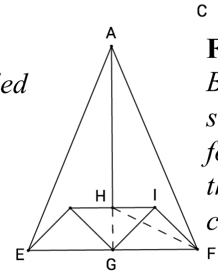

Figure 8: *Bisect auxiliary line segment $\overline{HF}$ by folding F onto H, then unfold to see crease $\overline{JK}$ in Figure 9*

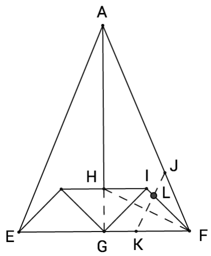

Figure 9:
Crease $\overline{JK}$ intersects $\overline{IF}$ at L. Bisect $\angle LGE$ (not drawn) by folding $\overline{GE}$ to overlap $\overline{GL}$ as shown in Figure 10

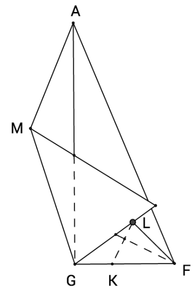

Figure 10:
$\overline{GM}$ is the bisector of $\angle LGE$, and it creates $\angle AGM$ which is half of one modular angle used to make the decagonal star

When ten modular pieces, as shown in Figure 10, are assembled, they make the following diagram:

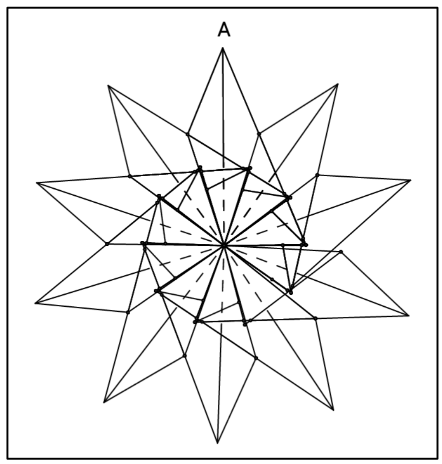

Figure 11: *Assembled decagonal star*

Calculations

Let's calculate the measure of $\angle AGM$ shown in figure 12 below. This is half of the angle from each of the modular pieces used to form the ten-point star. Without loss of generality, let the side length of the original square paper be 1. We can calculate the following:

1. Diagonal $AC = \sqrt{2}$ (see Figure 3)
2. Segments $C'G = GF = \sqrt{2} - 1$ (see Figure 6)
3. Segment $HG = \frac{\sqrt{2}-1}{2}$ (see Figure 7)
4. Using the Pythagorean Theorem in right triangle, $\triangle HGF$ from Figure 12 on the right, we can find $HF = \frac{\sqrt{15-10\sqrt{2}}}{2}$

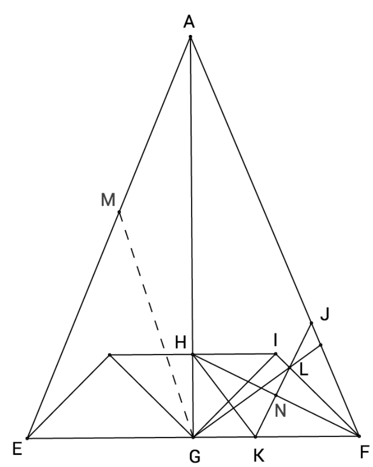

Figure 12

5. In Figure 13, $\overline{LK}$ is the perpendicular bisector of $\overline{HF}$. In $\triangle\, HGF$ and $\triangle\, KNF$, by using right triangle trigonometry, we can find the measures of $\angle HFG$ and $\angle NKF$ as follows:

$$m\angle HFG = tan^{-1}\left(\frac{HG}{GF}\right) = tan^{-1}\left(\frac{\frac{\sqrt{2}-1}{2}}{\sqrt{2}-1}\right)$$
$$= 26.57°$$

$$m\angle NKF = 90° - m\angle HFG = 63.43°$$

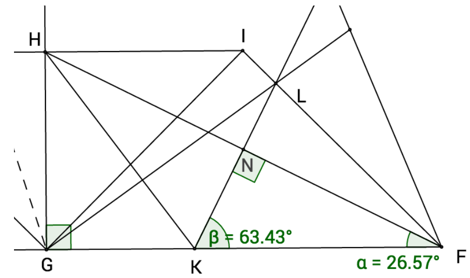

Figure 13

6. Since $\overline{LK}$ is the perpendicular bisector of $\overline{HF}$, we can find the measures of $\angle NKH$ and $\angle HKG$:

$$m\angle NKH = m\angle NKF = 63.43°$$
$$m\angle HKG = 180° - m\angle HKF = 53.13°$$

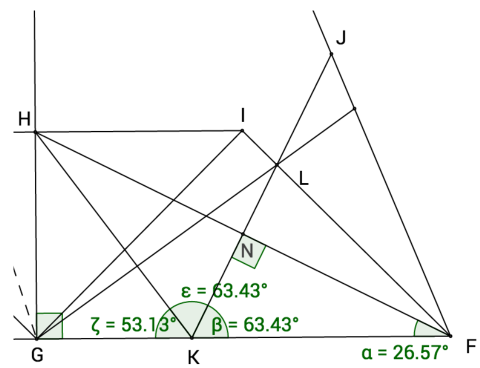

Figure 14

7. In right $\triangle\, HGK$, we can calculate GK and HK as follows:

$$GK = \frac{\sqrt{2}-1}{2\, tan\, 53.13°} = 0.1553$$

$$HK = \frac{\sqrt{2}-1}{2\, sin\, 53.13°} = 0.2589$$

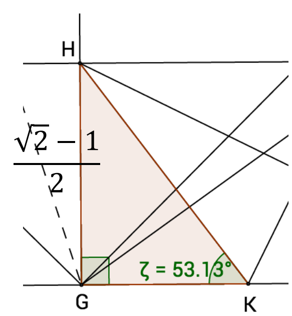

Figure 15

8. Since $GK = 0.1553$,
$KF = GF - GK = 0.2589$

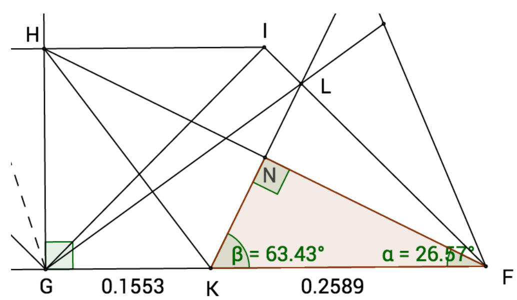

Figure 16

9. In $\triangle KLF$,
$m\angle KLF = 180° - 45° - 63.43° = 71.57°.$

Using the Law of Sines, we can find

$$KL = \frac{0.2589 \cdot sin\,45°}{sin\,71.57°} = 0.1930$$

$$LF = \frac{0.2589 \cdot sin\,63.43°}{sin\,71.57°} = 0.2441$$

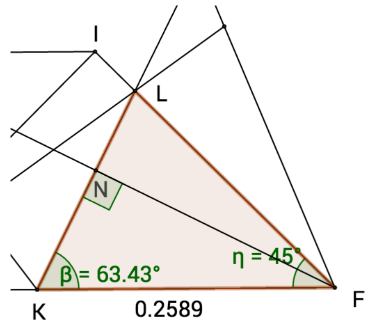

Figure 17

10. In $\triangle GLF$, we can calculate side GL by using the Law of Cosines:
$GL =$
$\sqrt{0.2441^2 + 0.4142^2 - 2 \cdot 0.2441 \cdot 0.4142 \cdot cos45°}$
$= 0.2969$

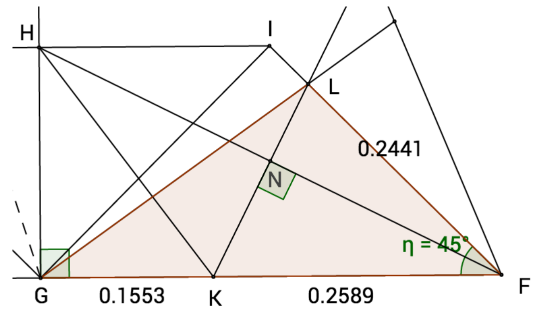

Figure 18

11. In $\triangle GLK$, we can use the Law of Sines to find
$m\angle LGK = sin^{-1}\left(\frac{0.1930 \cdot \sin 116.56°}{0.2969}\right)$
$= 35.54°.$

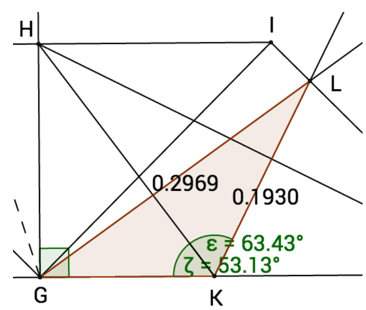

Figure 19

12. Since $\overline{MG}$ bisects $\angle LGE$,
$m\angle LGM = \frac{180° - 35.54°}{2} = 72.23°$
$m\angle LGH = 90° - 35.54° = 54.46°$

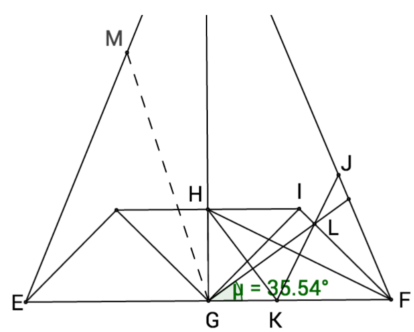

Figure 20

13. Finally,
    $m\angle HGM$ (or $\angle o$)
    $= m\angle LGM$(or $\angle \upsilon$) $- m\angle LGH$(or $\angle \xi$)
    $= 72.23° - 54.46°$
    $= 17.77°$

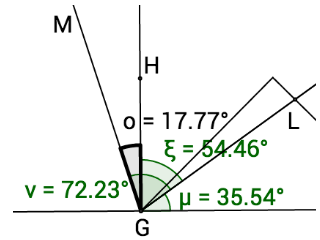

Figure 21

When ten modular pieces are assembled, we now have a better understanding of how they fit:

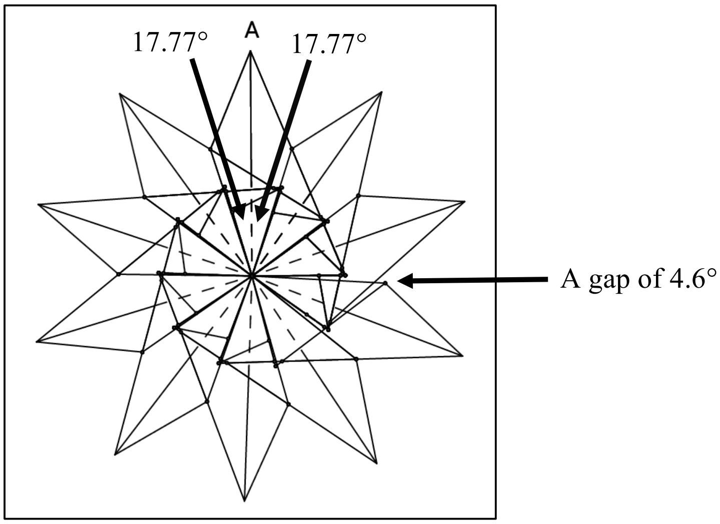

Figure 22: *Assembled decagonal star with measurements*

Conclusion

Each modular piece contributes an angle of 17.77° × 2, or 35.54°, for a total of 355.4° when ten modular pieces are assembled together. This is quite close to the 360° needed to make a complete circle. It allows little room to accommodate the thickness of the paper and inaccuracies in folding. These calculations show that this model has a tight fit and therefore accuracy in folding is important to properly assemble the pieces.

With a thoughtful selection of origami models, various origami activities can be readily accessible to students of all ages. Students can observe, hypothesize, and verify mathematical properties through folding. For students with more content knowledge, they can perform calculations by hand or constructions on a graphing tool to reinforce their understanding of mathematics. Through these activities, both mathematics and art are created and exercised on multiple levels and in multiple forms. From the beginning to the end, we practice the art of posing good questions, the art of problem solving, the art in the physical folding, and the art in the final product, the origami stars.

References

[1] Joy Hsiao, "Finding Fifths in Origami," *Mathematics Teacher*, 109(1): 71-75, 2015.
[2] Tomoko Fuse, *Home Decorating with Origami*, Japan Publications Trading Company, 2000.