Listening to the Logistic Map

Andrea Capozucca[1]
University of Urbino
Piazza della Repubblica 13,
61029 Urbino, ITALY

Marco Fermani
Musician and Composer
Via Regina Margherita 160,
62018 Potenza Picena, ITALY

Simone Giorgini
Musician and Composer
Via Damiano Chiesa 11,
60021 Camerano, ITALY

Abstract

A logistic map is one of the simplest nonlinear dynamical systems that clearly exhibits a route to chaos. In this paper, we present the creation of a musical piece based on a logistic map. We explored the evolution of the logistic map using a free sonification program called "Music2Chaos". We divided the one-dimensional interval [0,1] into eleven equal parts, and associated a note of a pentatonic scale to each segment. Every time an iteration took place a corresponding orbit was generated and translated into a sequence of notes. Then, we built up the main theme of the musical piece by choosing the sequences that best express the surprising array of dynamical behaviors of the logistic map, and assembled them according to harmonic rules of music. Time and musical arrangements were chosen to best fit and define the complete song structure.

Introduction

Difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behavior, from stable equilibrium points to cyclic patterns and even "chaotic" behavior, a regime which, although fully deterministic, is in many respect indistinguishable from a random process. The behavior of such equations can be studied by manipulating and processing large amounts of data via a computer. The results of such calculations often have to be transformed into visual information to be easily accessible to the human analyzer, and provide a clear vision of the dynamical system's global evolution (e.g., bifurcation diagrams or strange attractors). If we are able to "see" the behavior of such equations, why could we not "hear" them?

Converting the logistic map to music

The logistic map is, perhaps, the simplest example of how a nonlinear dynamic equation can give rise to very complex behavior. Initially introduced as a mathematical model of population growth, it rapidly found applications in diverse areas like mathematical biology, biometry, demography, condensed matter, physics, econophysics, and computation [1]. The logistic map function is defined as

$$x_{n+1} = kx_n(1 - x_n) \equiv f_k(x) \,, \qquad (1)$$

where k is a model-dependent exogenous parameter and x_n is the population in the nth-period, scaled so that its value fits the interval $[0,1]$.

We used the outputs of the logistic map to generate music. The form of the piece is always subject to our artistic aims as composers taking care not to manipulate the raw material in an "extensive and idiosyncratic" way [2]. It is important that the music remains compelling to an audience and, unlike Babbitt [3], we must care if they listen.

Choosing a musical scale. We chose the pentatonic scale, first of all, for its universality. Humans show a natural predisposition towards the pentatonic scale. Pentatonic scales are found all over the world, from

[1] Email: `andrea.capozucca@uniurb.it`

Pre-Columbian music to African music, Irish folk, Chinese music and the 35,000 year old bone flute from Hohle Fels. Furthermore, music performed using pentatonic scales does not have musical dissonances and always sounds pleasant. Several anthropologists have reported that we are innately prone to appreciate these melodic sounds and our ears prefer the simple mathematical ratios between frequencies [4]. We decided to use the A minor pentatonic scale (A-C-D-E-G), which is considered a gapped blues scale and comprises the same tones as the C major pentatonic, because the arrangement of notes in minor scales plays a large role in determining the mood of music.

Music2Chaos (M2C). This is a free sonification program[2] that allowed us to transform a sequence of numbers (the orbit generated from the iteration of the logistic map function f_k starting from an initial seed) into a sequence of notes visible within a red frame, easy to listen to and exportable via a midi file.
As shown in Fig.1, in the upper part of the M2C program window, we can choose the Midi Device, tempo in beats per minute (bpm), number of tracks to play simultaneously, number of iterations to calculate, the parameter k value to start the simulation, to play the sequence and to save it in a specific format. On the right, we have the red frame inside which the sequences are stored according to the number of iterations (we can choose the value of the initial seed $x_0 \in [0,1]$). On the left, the array of notes chosen for the simulation are shown: two A minor pentatonic scales starting with A3[3] and ending with A5. In this case, we divided the [0,1] interval into eleven equal parts and mapped one-to-one to a single note in the array. When the logistic map produces certain values, the program sets the corresponding note to the numeric output according to the interval partition (e.g, $x_2 = 0.70875$ corresponds to D5). After several iterations, it is possible to visualize the orbit obtained and listen to the musical sequence.

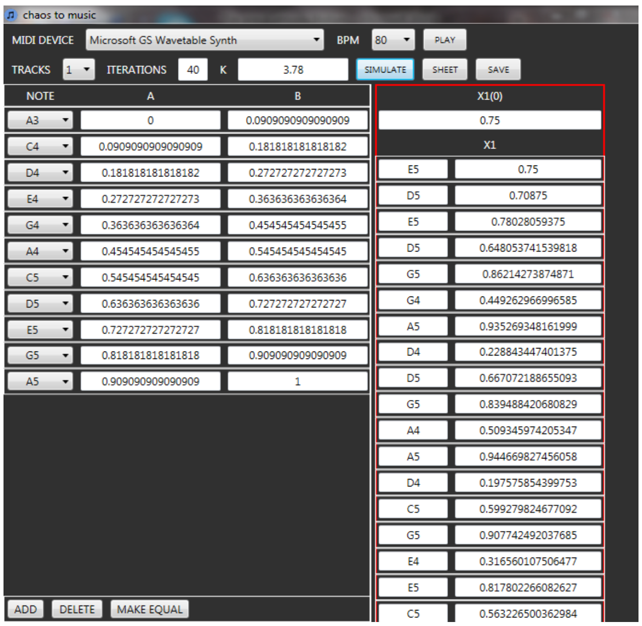

Figure 1: *A snapshot of "Music2Chaos" interface window.*

Choosing a starting note. We wanted to give a sad and weird feeling to the musical piece. So we chose A3 as the starting note for every simulation regardless of parameter k value. For that to happen, we chose as the initial seed for every simulation a value within the range $[0,0.090909]$ corresponding to the A3 note. $x_0 = 0.078$ was the choice that leads to music the authors liked. In addition, starting from A3 allowed us to get ever-increasing note sequences concerning stable fixed-point behavior of the logistic map ($1 < k < 3$).

Instruments. We decided to use a contrabass, an accordion and a synthesizer because their tones best fit the desired mood of the musical piece. Along the track, a musical interplay developed that created movement and flow within the song and musically described different feelings and settings for every system state. Moreover, we used 8 different beat-box types for the rhythm section, to increasingly emphasize the sound picture we had in mind. Beat-box is present for the duration of the song, except at some crucial steps.

[2] "Music2Chaos" has been created for the purpouse by Andrea Capozucca and Diego Zallocco.
[3] A3 is the 3rd octave regular A in a standard 8-octave keyboard.

K-sequence generation and final assembling

We divided the musical piece into three main sections: a stable fixed-point (SFP) section, a periodic even cycle (PEC) section and a chaotic (CPW) section with periodic (odd cycle) windows inside. Using M2C, we generated several orbits choosing appropriate values for k. The tempo was set at 120 bpm.

k	Note sequences	Fixed point x
0.8573	A3 A3 A3 A3 A3 **A3** ...	0
1.19	A3 A3 C4 C4 C4 C4 **C4** ...	0.15966386...
1.3687	A3 C4 C4 C4 C4 D4 **D4** ...	0.26937970...
1.5224	A3 C4 C4 D4 D4 E4 **E4** ...	0.34314241...
1.7957	A3 C4 D4 E4 G4 G4 **G4** ...	0.44311410...
1.9042	A3 C4 D4 E4 G4 A4 **A4** ...	0.47484507...
2.2271	A3 C4 E4 A4 C5 C5 **C5** ...	0.54954954...
2.7804	A3 D4 G4 D5 C5 D5 C5 D5 C5 **D5** ...	0.64028776...

SFP section $(0 \leq k < 3)$

We built this section assembling the note sequences shown in the table on the left, comprised of five beats each. The duration of every single note was established following two simple rules:
1) the first part of the sequence was emphasized to highlight the speed of the convergence to stability;
2) a standard duration (2/4) was assigned to notes corresponding to a fixed-point value.

In this section, all the note sequences in the table above are played by the contrabass, whereas the accordion just plays a musical accompaniment, in addition performing the breath beat-box. Note duration can be freely chosen to improve the musicality.

k	Note sequences	2^n-cycles
3.0214	A3 ... **D5 C5** ...	2-cycle
3.089	A3 ... **E5 C5** ...	
3.1987	A3 ... **E5 A4** ...	
3.2845	A3 ... **G5 A4** ...	
3.4452	A3 ... **G5 G4** ...	
3.4972	A3 ... **G5 G4 G5 A4**...	4-cycle
3.5523	A3 ... **G5 G4 G5 A4 G5 E4 E5 A4** ...	8-cycle

PEC section $(3 < k < 3.569946)$

To better express the periodic behavior, we purposely chose to cut the first part of the sequences and take into account only the notes contained in the cycles[4] (see the bold notes on the left table). We also decided to match each cycle with a musical mood (2-cycle/waltz, 4-cycle/funk, 8-cycle/reggae) to convey different feelings to each system state.

Unfortunately, we cannot "observe" higher periodicity (like a 16-period or higher) with this setup because of the discretization used for the interval $[0,1]$. In order to increase the periodicity, we would have to divide the $[0,1]$ interval into more than eleven parts, which would create too many notes to handle and, therefore, less musical enjoyment.

In this section, we disregarded the beginning notes of every sequence taking into account only the bold notes in the table above, corresponding to that stable cycle. Furthermore, we introduced an interesting role reversal between instruments to better express these periodic behaviors. In 2-cycle/waltz and 8-cycle/reggae the contrabass plays the notes, whereas the accordion is part of the arrangement. In the 4-cycle/funk, the role of the instruments was interchanged.

[4] We generate more 2-cycle sequences than 4-cycles or 8-cycles to better show what happen after the first bifurcation point for $k = 3$. After this point, the two branches of the bifurcation diagram tends to broaden due to different stable point values.

k	Note sequences	Behavior
3.65764	A3 D4 A4 E5 D5 E5 D5 E5 C5 G5 E4 G5 A4 A5 E4 E5 D5 G5 ...	Chaotic
3.70164	A3 D4 D5 E5 D5 C5 A5 E4 E5 D5 **G5 A4 A5 D4 D5 E5 D5**...	7-cycle
3.72048	A3 D4 E5 E5 D5 E5 D5 E5 C5 A5 E4 E5 D5 E5 C5 G5 G4...	Chaotic
3.7383	A3 D4 E5 E5 E5 D5 E5 C5 G5 G4 G5 E4 **G5 A4 A5 D4 D5** ...	5-cycle
3.7964	A3 E4 E5 D5 E5 C5 G5 G4 G5 E4 G5 A4 A5 D4 C5 A5 D4...	Chaotic
3.82843	A3 E4 E5 D5 E5 C5 A5 D4 D5 G5 A4 A5 **C4 A4 A5** ...	3-cycle
3.9375	A3 E4 E5 C5 A5 D4 D5 G5 G4 A5 C4 E4 G5 G4 A5 A3 A3...	Chaotic

CPW section $(3.569946 < k \leq 4)$

We assigned a standard duration (1/4) to chaotic notes and a blues/swing mood to the section with the initial setup. A synthesizer played the chaotic sequence and instruments played a musical improvisation over the track. In this section, the beat-box is 1/4 ahead of the beat to avoid overlapping contrabass. Moreover, a periodic window appears every 12 beats: first 7-cycle, then 5-cycle and finally 3-cycle (see left table). The musical piece ends by speeding up the beat until it becomes white noise.

In this section, chaotic sequences played by the synthesizer exhibit endless variations within the limits of the scales utilized. Instrumental improvisation and composer add-ons enhance this underlying structure. Moreover, the piece presents three periodic windows at regular intervals representing the most significant periodic behaviors in the chaotic region.

Conclusion

Turning the logistic map into a musical piece that makes sense is no trivial task, especially regarding its chaotic behavior. Finding a correct subdivision of $[0,1]$ intervals and, consequently, an appropriate array of notes is not difficult, but is also not guaranteed. Furthermore, in SFP and PEC sections, we have to face a problem of orbit quick convergence that forces us to carefully choose specific k values to generate meaningful sequences. In CPW section, the use of the A minor pentatonic scale proved to be useful to blend a compelling chaotic background structure with the whole arrangement. We agree with Truax's statement that "the musicality may reside in the musical knowledge of the mapper than in the source function"[5]. But there is an inevitable level of error in our attempt at describing a logistic map through music, either in terms of rigor or in terms of communicability. Finally, to better understand the musical piece as a description of a logistic map, behaviors could be displayed with a bifurcation diagram with appropriate text and graphic representations to complement its listening. It could also be a different way to present, in a classroom, the practical and creative applications of non-linear dynamic systems.

References

[1] M. Ausloos, M. Dirickx (eds.). *The Logistic Map and the Route to Chaos: From the Beginnings to Modern Applications*. Springer, New York (2006)

[2] R. A. Bidlack. *Chaotic systems as simple (but complex) compositional algorithms.* Computer Music Journal (1992), p. 33-47.

[3] M. Babbitt. *Who Cares if You Listen?*. High Fidelity (Feb. 1958)

[4] F. J. Ballesteros. *E.T. Talk: How We Will Communicate With Intelligent Life on Other Worlds*. Springer (2010), p. 160.

[5] B. Truax. *Chaotic non-linear systems and digital synthesis: An explanatory study.* Proceedings of the International Computer Music Conference (ICMC 1990), p. 100-103.