Conics from Polygons: The Chord Ratio Construction

Scott Vorthmann
vZome author
20541 Wardell Rd
Saratoga CA 95070, USA
`scott@vorthmann.org`

David Hall
vZome collaborator
494 Bouldercrest Dr.
Marietta GA 30064, USA
`david.42.hall@gmail.com`

David Richter
Associate Professor
Department of Mathematics, MS 5248
Western Michigan University
Kalamazoo, MI 49008-5248, USA
`david.richter@wmu.edu`

Abstract

We present a new technique for constructing conic sections and quadric surfaces, or more specifically, points on those curves and surfaces. The technique, called the chord ratio construction, takes a single numeric parameter and three input points, producing a fourth point. We show how the parameter value determines what curve or surface contains the points produced by iterating the construction. Along the way, we present some artworks that both inspired these results, and were inspired by them.

Introduction

Conic sections (ellipses, parabolas, and hyperbolas) and quadric surfaces (ellipsoids, paraboloids, and hyperboloids) are normally the domain of analytic geometry, and would seem to have little to do with regular polygons. However, the results we present here grew out of our work generating *affine regular* polygons in vZome, a geometry software application. Experimenting with a new affine regular heptagon command in vZome, Hall discovered that such heptagons joined seamlessly to form an unbounded polyhedral surface, with the vertices apparently lying on half of a two-sheet hyperboloid. That discovery inspired some artworks, and triggered our investigation into the mathematics of what we now call the *chord ratio construction*. As we will show, the construction can be applied iteratively in two dimensions to generate points on any type of conic section, with the type depending solely on the chord ratio applied. Furthermore, we will show how the construction can be applied in three dimensions to produce a continuous range of quadric surfaces. Again, the type of quadric surface is dictated by the chord ratio in use.

Affine Regular Polygons

An *affine regular polygon* is the result of applying an affine transformation to a regular polygon. An affine transformation is a linear transformation that does not preserve distances and angles in general, but does preserve parallel lines, and the ratios of distances along those lines. Figure 1 shows a regular heptagon and an affine regular heptagon.

An affine pentagon command has been available in vZome for many years, producing an affine regular pentagon when applied to any three non-collinear points. When applied to *four* non-coplanar points, three at a time, then repeated at all subsequently generated points, the result is an affine regular dodecahedron, as seen in Figure 2.

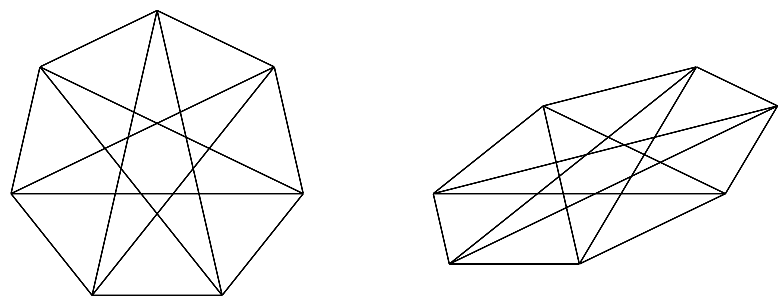

Figure 1: *A regular heptagon and an affine regular heptagon. In both cases, each diagonal is parallel to one edge, and longer than that edge by a factor* ≈ 2.247*.*

Considering that a normal regular polygon can be inscribed in a circle, and an affine transformation of a circle yields an ellipse, clearly the vertices of each face of the affine dodecahedron all lie on an ellipse. Similarly, the vertices of the entire object must lie on an ellipsoid, an affine-transformed sphere. These are the first hints of the relationship we have discovered between polygons, conic sections, and quadric surfaces.

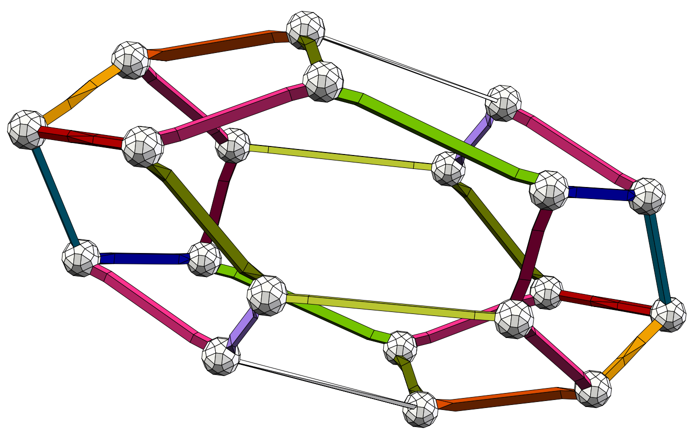

Figure 2: *An affine regular dodecahedron, as realized in vZome.*

Mathematics Inspires Art Inspires Mathematics

Every regular polygon generates an algebraic field based on the lengths of its diagonals, as has been nicely explained by Steinbach [7] [8] and Kappraff [5]. For example, the regular heptagon with unit edge length has two diagonals, with lengths $\rho \approx 1.802$ and $\sigma \approx 2.247$. The generated field $\mathbb{Q}[\rho, \sigma]$ consists of numbers of the form $a + b\rho + c\sigma$, for $a, b, c \in \mathbb{Q}$.

vZome is based on the mathematics of such algebraic fields, taking advantage of them by using only exact integer arithmetic until the point of rendering, when the irrationals like ρ and σ are finally used to compute floating-point values. Vorthmann experimented with support for the heptagon field years ago, and Hall revived that work recently. Hall realized that he could implement an affine heptagon command as a simple modification to the existing affine pentagon command. This was the genesis of the generalization that became the chord ratio construction.

Experimenting with this new capability, Hall discovered that affine heptagons could be joined seamlessly to form an unbounded polyhedral surface. The surface seemed to be a polygonal faceting of one sheet of a two-sheet hyperboloid. In fact, this could be done with affine regular N-gons for any $N \geq 6$. This inspired him to produce several works of digital art. They can all be viewed online, and manipulated with interactive 3D controls, at `https://sketchfab.com/davidhall/collections/hyperbolic-tilings`.

Figure 3: *Hyperbolic Heptagonal Hydrangea*

Figure 3 shows Hall's "Hyperbolic Heptagonal Hydrangea" [2], the original and simplest example. In this piece, Hall started with three mutually orthogonal edges, and generated three affine heptagons from them. The process is repeated by generating a new affine heptagon in the v-shaped gap between existing facets. After three generations of this, each generation forming a ring of facets, a faceted hyperboloid shape clearly emerges, which Hall calls a *hyperbolohedron*.

Another piece, "4x2 Heptagonal Hydrangea" [3], shown in Figure 4, has four "layers" of concentric hyperbolohedra, include one that is completely planar. Each layer is generated by its own initial set of 3 edges, all sharing the same common vertex. Since all of the initial sets of edges have threefold rotational symmetry around their common vertex, the resulting hyperbolohedra also share that threefold rotational symmetry around a common axis.

The progression culminates in Hall's "4x5 Heptagonal Dandelion" [4], shown in Figure 5, which again features four layers, but here he has iterated the generation of affine heptagons further, going out five generations in each layer. The affine heptagons quickly become so elongated as to be nearly unrecognizable.

These pieces played a pivotal role in our collaboration on the mathematical results we describe below. The pieces were somewhat surprising; why should constructing these heptagons tile a surface without gaps or

Figure 4: *4x2 Heptagonal Hydrangea*

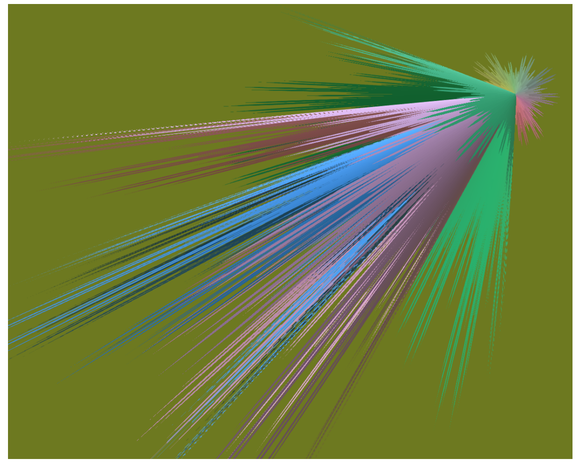

Figure 5: *4x5 Heptagonal Dandelion*

overlaps? At first glance the 3D shapes approximated by the polygon facets appeared to be quadric surfaces, but were they? Answering these questions required applying some mathematical rigor to the constructions, as we will describe below.

Generalizing: the Chord Ratio Construction

When Hall implemented his affine heptagon command, he did it in a way that generalizes as the *chord ratio construction*, illustrated in Figures 6 and 7. The construction requires three points A, B, and C and a *chord ratio* R, and generates one additional point D, such that $AD \parallel BC$, and $|AD|/|BC| = R$. The degenerate case where A, B and C are collinear results in D also being collinear. Obviously, this is not simply a classic ruler-and-compass construction; it requires a proportional divider. Notice also that the construction can be applied with a reverse orientation (switching the labels of points A and C); thus the initial three points actually determine two new points.

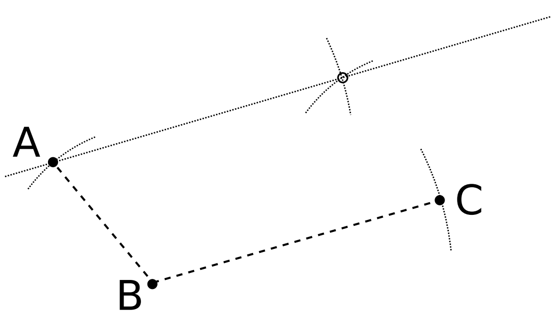

Figure 6: *Step 1, construct parallel to BC.*

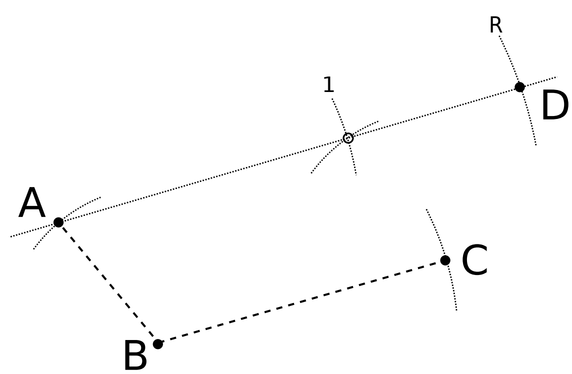

Figure 7: *Step 2, construct D using ratio R*

We have defined the construction in terms of vertices, but we often think of it in terms of the connecting line segments, particularly when they connect up to form a polygon. Hall's trick with the heptagon was to apply the chord ratio construction iteratively, using points B, C and D from each iteration as inputs into the next iteration, to generate all vertices and edges of the affine regular heptagon.

Hall was curious about the relationship between the chord ratio R and the number of sides of the polygon produced for certain values of R, so he built an interactive notebook in Geometer's Sketchpad , shown in Figures 8 and 9.

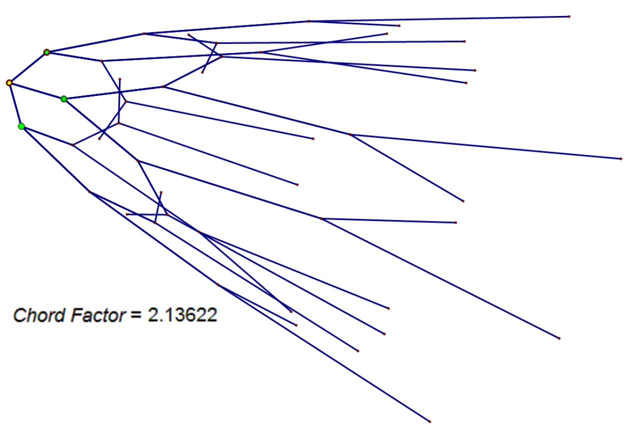

Figure 8: *Typical model with* $2 < R < 3$.

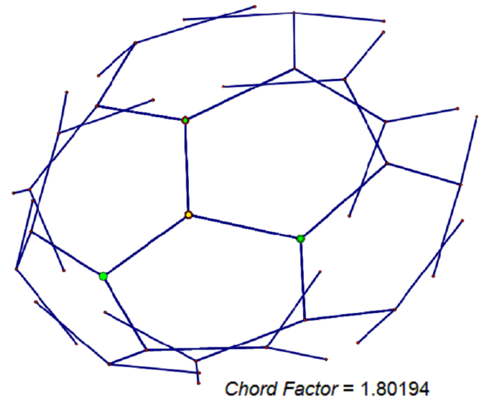

Figure 9: *Typical model with* $1 < R < 2$.

In the Geometer's Sketchpad notebook, he manually iterated the chord ratio construction several levels out from an initial set of four vertices. The notebook allows free movement of these four initial points, as well as a slider for continuous adjustment of the chord ratio from 0 to 3, using the same chord factor for all vertices in the model. The result generally looks something like a tree, but that depends strongly on the value of the chord ratio. Additionally, moving the four initial points allows us to simulate a "side view" of a 3-dimensional configuration of the input edges as shown in Figure 8.

Hall found that the iterated construction of edges produces an affine regular polygon when $R = sin(3 * 180°/N)/sin(180°/N)$, where N is an integer. When $N = 3, 4$ or 5, we find that $R = 0$, $R = 1$ or $R \approx 1.618$ respectively. These affine polygon faces join to form affine versions of familiar regular polyhedra. For example, using the golden ratio, $R \approx 1.618$, the model produces an affine dodecahedron similar to Figure 2.

However, the non-convergent, infinite-edged "faces" are actually more interesting. For most values of R, the iterated construction does *not* produce a finite set of vertices. For $0 < R < 3$, these faces are always elliptical, in the sense that all of the vertices lie on a single ellipse. For $R > 3$, the sequence diverges, and forms an hyperbola. This implies that $R = 3$ produces parabolic faces. We will prove these assertions below.

Even when the faces remain elliptical in shape, the three-dimensional surface that they lie on can open up and diverge to infinity. As expected, for $R \approx 2.247$ (the length of the longer diagonal of a heptagon having an edge length of 1), the faces are affine heptagons, but the infinite polyhedron they form has vertices that lie on half of an hyperboloid of two sheets, just as Hall demonstrated in his artworks. Here, again, there must be a threshold value, at which the envelope of the vertices transitions from ellipsoid to hyperboloid. That threshold, where the envelope is a paraboloid, is when $R = 2$. For $R > 2$, the envelope is an hyperboloid. We formalize (somewhat) these assertions in the following theorems, whose proofs we will provide below.

Theorem 1 (Chord Ratio Conics)**.** *The iterated chord ratio, applied to three distinct points in the plane, produces points that lie in a conic section, with the type of conic solely determined by the chord ratio* R *used consistently throughout.*

Theorem 2 (Chord Ratio Quadrics)**.** *The iterated chord ratio, applied to four distinct points in* $\mathbb{R}^3$, *produces points that lie on a quadric surface, with the type of quadric solely determined by the chord ratio* R *used consistently throughout.*

Proving the Chord Ratio Conics Theorem

For values of R between zero and three, it is easy to show that the iterated chord ratio construction starting from three points produces points on an ellipse. Clearly, from non-collinear initial edges AB and BC, the construction generates a fourth point that completes a trapezoid $ABCD$. Figure 10 demonstrates the fact that any such trapezoid can be transformed to a 3-sides-equal trapezoid by a composition of affine transformations.

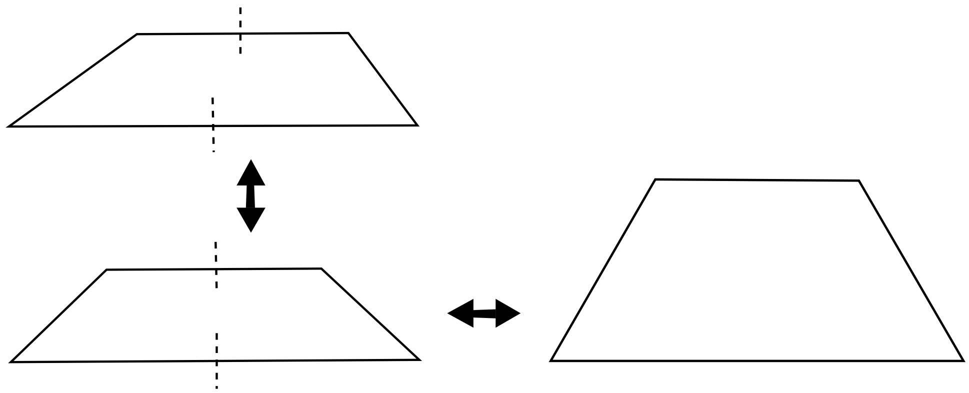

Figure 10: *Any trapezoid can be transformed to a 3-sides-equal trapezoid by applying a pair of affine transformations, a shear and a stretch. The inverse is also true.*

Furthermore, Figure 11 demonstrates that every 3-sides-equal trapezoid can be inscribed in a circle. Now we can apply the inverse affine transformations, turning the circle into an ellipse, and restoring the original trapezoid shape. The figure also demonstrates why this proof cannot be applied for values of $R > 3$.

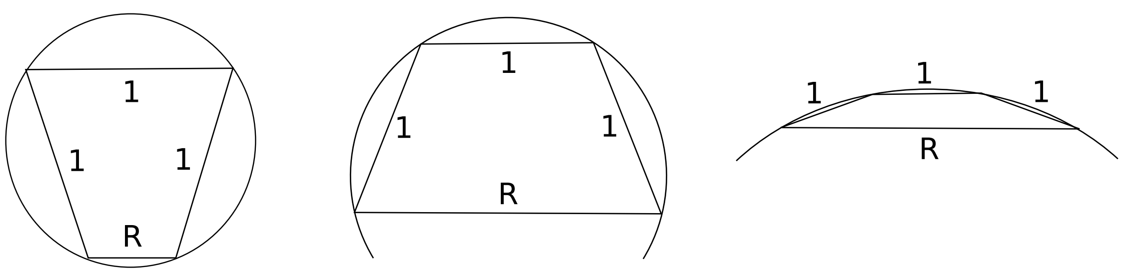

Figure 11: *Any 3-sides-equal trapezoid can be inscribed in a circle. The length R of the unequal side varies between zero and three times the length of the other sides.*

To prove our theorem more completely, we focus on a simplified case, where the initial three points are $(0, 0)$, $(0, 1)$, and $(1, 0)$ in the cartesian plane. The first pair of generated points are $(0, R)$ and $(R, 0)$. It is a straightforward exercise to derive the equation of a conic section containing these five points, since five points are sufficient to uniquely determine a conic section:

$$x^2 + y^2 + (1 - R)xy - x - y = 0$$

The discriminant for that equation is $R^2 - 2R - 3$. It clearly has a value of zero when $R = 3$, indicating that the conic section is a parabola. A value of $R > 3$ yields a discriminant greater than zero, indicating an hyperbola, and $0 < R < 3$ similarly indicates an ellipse.

Since any other initial triplet of non-collinear points can be mapped to these three by an affine transformation, and affine transformations preserve the class of a conic section, we have proven Theorem 1.

Proving the Chord Ratio Quadrics Theorem

A similar approach can be taken in the case of three initial edges, by starting with four points: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Applying the chord ratio construction to each possible combination, we obtain six more points: $(0,1,R)$, $(0,R,1)$, $(1,R,0)$, $(R,1,0)$, $(1,0,R)$, and $(R,0,1)$. Since ten points determine a quadric surface, it is easy to verify that these points all lie on the surface whose equation is

$$x^2 + y^2 + z^2 + (R-1)xy + (R-1)yz + (R-1)xz - x - y - z = 0$$

The classification of this quadric surface can be accomplished using standard techniques [1], based on representing the equation as a matrix equation, and computing the determinants of the matrix and its three-by-three upper-left submatrix, and finding the eigenvalues of both. Some tedious but straightforward algebra (solving a cubic characteristic polynomial) yields these eigenvalues:

$$\lambda_1 = \lambda_2 = \frac{R+1}{2}, \lambda_3 = 2 - R$$

For $R > 0$, the first two eigenvalues must be always positive. The third eigenvalue vanishes when $R = 2$, and is negative for $R > 2$. Those facts confirm the results in the following table, characterizing both the conic section ”faces” and and the quadric surface overall.

range	”face” conic section	”solid” quadric surface
$0 < R < 2$	ellipse	ellipsoid
$R = 2$	ellipse	elliptic paraboloid
$2 < R < 3$	ellipse	hyperboloid of two sheets (half)
$R = 3$	parabola	hyperboloid of two sheets (half)
$3 < R$	hyperbola	hyperboloid of two sheets (half)

Although we proved the results for a specific, simplified case, the result generalizes as did our proof for conic sections, above. Given a collection of points and edges generated from four arbitrary non-coplanar points, we can define an affine transformation that maps those initial points, and thus the entire construction, to our simplified case. Again, the affine transformation, and its inverse, preserves the class of quadric surface containing all the constructed points.

Mathematics Inspires More Art

Vorthmann has produced a physical sculpture embodying the mathematical results we have described above. The sculpture, called ”Polyhedral Paraboloid”, is constructed from Zometool and 3D-printed extensions to Zometool. It consists of a lattice of connectors and struts, joined in a collection of affine regular hexagons. Therefore, according to our mathematical results, all the connectors in each of the two layers lie on an elliptic paraboloid.

Conclusion

We have demonstrated that the iterated chord ratio construction can construct points along any continuous conic section, when applied to three initial non-collinear points, just by varying the chord ratio. We have

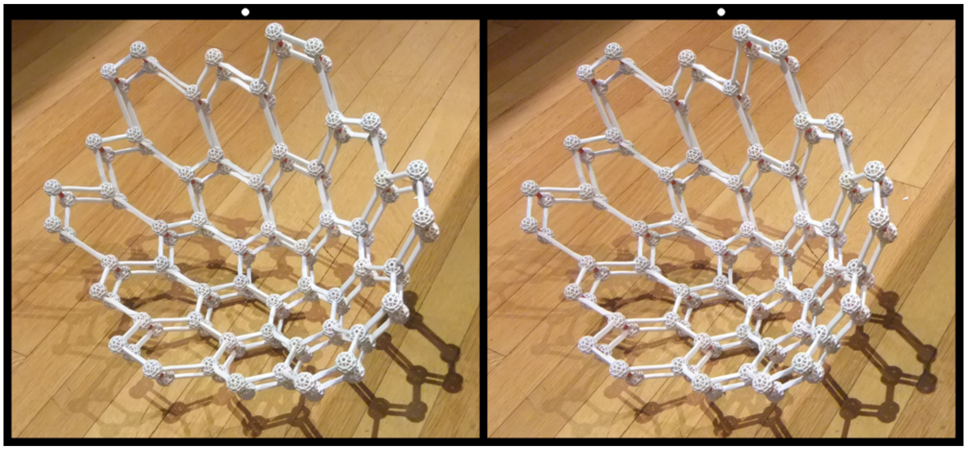

Figure 12: *Polyhedral Paraboloid, by Scott Vorthmann. Wall-eyed stereo pair.*

further demonstrated that the same construction, applied to four initial non-coplanar points, can generate an arbitrary number of additional points lying on a continuous quadric surface. We believe this is a new, if limited, construction technique for conic sections and quadric surfaces, in spite of the fact that many conic constructions have been known since Appolonius [6]. We would therefore appreciate any leads regarding previous publications of this construction technique in any of the classical literature.

There are other questions to explore, such as aspects of the infinite tiled surfaces produced when the faces are regular polygons with seven or more sides, as we see in Hall's and Vorthmann's artwork. Richter has started to explore these aspects in terms of reflection groups, using an atypical formulation of "reflection".

As this document was being prepared for publication, Hall experimented with negative values of R. He found that the points generated by the chord ratio construction continue to fall on an ellipse, until $R = -1$, when the ellipse stretched into a pair of parallel lines. For $R < -1$, the generated points alternate between two halves of an hyperbola. This is all consistent with the discriminant stated above, which has a zero at $R = -1$, producing a parabola, although a degenerate one. Hall's findings will be developed more fully in a supplemental paper, available from the authors.

References

[1] A. Dresden, *Solid Analytical Geometry and Determinants*, John Wiley and Sons (1948).

[2] D. Hall, "Hyperbolic Heptagonal Hydrangea" `https://skfb.ly/RTQr`

[3] D. Hall, "4x2 Heptagonal Hydrangea" `https://skfb.ly/SAVw`

[4] D. Hall, "4x5 Heptagonal Dandelion" `https://skfb.ly/SAXX`

[5] J. Kappraff and G.W. Adamson, "A Unified Theory of Proportion", in *Conference Proceedings of ISAMA-Bridges 2003*, J. Barrallo, et. al. editors (2003).

[6] P. Kunkel, "Conics of Appolonius" `http://whistleralley.com/conics/conica/`

[7] P. Steinbach, "Golden Fields: A Case for the Heptagon", *Mathematics Magazine* , 70(1)(1997).

[8] P. Steinbach, "Sections Beyond Golden", in *Bridges: 2000* , ed. R. Sarhangi, Winfield (KS): Central Plains Books (2000).