Kissing Rings, Bracelets, Roses and Canadian Magnetic Coins: Circle packing with Ferrite Block Magnets and Magnetic Sheet

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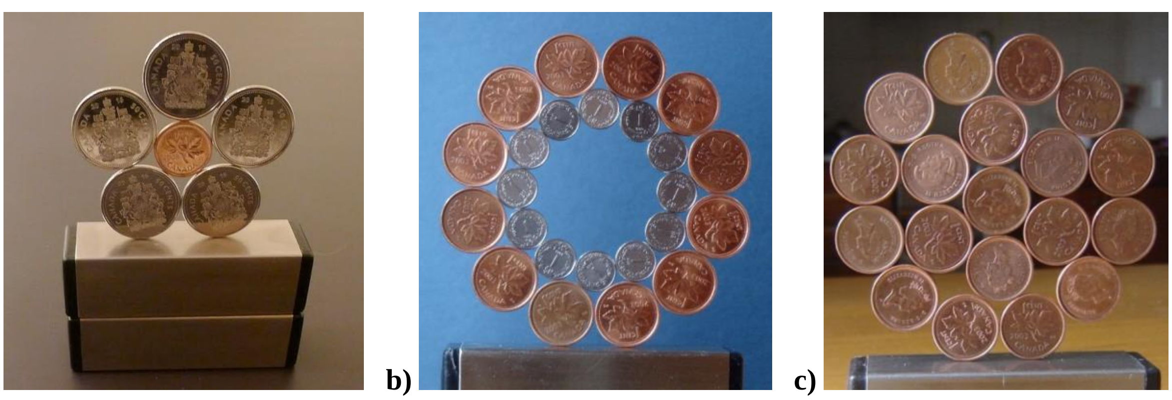
Abstract

The Royal Canadian Mint has a long history of producing high quality coins made of pure nickel or plated steel. As such, the coins are ferromagnetic enabling them to be fixed to a surface, or suspended or stacked on their rims via magnetism. The background to such use was described at the 2015 Bridges Baltimore Conference. Here we provide further artistic examples of near-close-packed coin arrays constructed using safe ferrite block magnets or magnetic sheet. In many cases only coins of Canada are present. Where Canadian coins of a suitable diameter are unavailable, coins of the UK, the European Union, Argentina, Brazil or Uruguay are included. The emphasis of this paper is on the geometry of bracelets and roses, formed from two or more connected rings, and how well we can approximate a perfect close-packing given the restricted diameters of the found objects, coins, used in our art.

Introduction

The use of coins in circle packing is not new [1, 2]. The use of magnetic coins in particular was outlined at the 2015 Bridges Conference [3]. In this paper, the focus is on aesthetically pleasing circular arrays made using mainly Canadian magnetic coins. The smaller arrays have been constructed, either standing or hanging, in magnetic fields provided by robustly covered common ferrite rectangular block magnets as in Figure 1. Larger arrays are attached to magnetic sheet of the type widely used for fridge magnets and whiteboard accessories.



a) b) c)

Figure 1: *Standing close fitting arrays. a) Center one cent and surrounding 50 cent Canadian coins: the diameter ratio a/b = 19.17/27.1= 0.705 is very near to the iron ratio 0.701 [3]. b) Standing kissing bracelet of 12 Canada 1 cent coins and 12 Uruguay (1981) 1 peso coins. c) Patagonia Rose analogue [2, 3] of Canada 1 cent coins. In above figures, the backgrounds and coins are not in contact.*

Magnetic Coins of Canada

Early Canadian coins were made of silver or copper. In 1922 the first 5 cent coin made of almost pure nickel was produced. From 1968-2000 most Canadian coins were made of pure nickel. Two particular exceptions were the 1 cent coin which continued to be made of copper and the 5 cent coin which, between 1982 and 1999, was made from a non-magnetic cupro-nickel alloy [4].

In the 1940s and early 1950s, some coins were temporarily made of plated steel. In 2000, all the 5, 10, 25 and 50 cent coins, and in 2012, both the 1 dollar and the outer ring of the 2 dollar coins began to be made of plated steel. Between 2000 and 2012, many 1 cent coins were also made of plated steel. The rest were made of plated zinc and consequently are non-magnetic.

All the 2013-2016 circulating Canadian coins are magnetic. The diameters are given (Table 1). For comparison, values of other magnetic coins often used in packing studies are also included. More examples can be found in the Magnetic Coin directory [5].

Ca 1 Dollar (to 1987)	32.1	**Ca 1 Dollar (1987 on)**	26.5	**Ca 5 cents**	21.3	Ar 5 cents	18.2
Ca 2 Dollar	28.0	UK 2p	26.0	UK 1p	21.2	**Ca 10 cents**	18.0
Ca 50 cents	27.1	UK 10p	24.4	US 1 cent (1943)	20.3	UK 5p	18.0
Br 1 Real	27.0	**Ca 25 cents**	23.9	**Ca 1 cent (to 2012)**	19.1	EU 1 cent	16.3
Ca 1 Dollar (1987)	26.7	EU 5 cents	21.3	EU 2 cents	18.8	Ur 1 peso (1981)	12.0

Table 1: Diameters (mm) of magnetic coins of Canada and other countries.

Kissing Circles and Kissing Bracelets

In 1936, the radiation chemist and Nobel Laureate, Frederick Soddy, published his poem *The Kiss Precise* in the 1936 Nature journal [6]. Inspired by the work of Descartes, its opening lines read:

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.

Today, the "kissing" of circles is a familiar concept not only in mathematics but also to those interested in their role in nature and industry as well as in art and culture [7, 8, 9, 10]. *Kissing circles* are circles that meet tangentially, intersecting at only one point. We define an *n-circle ring* as an array of *n* circles all having the same diameter and arranged so that the centers of the circles are concyclic. Each circle in a ring is tangent to two others. When two rings are close-packed, we call the combined circular array a *kissing bracelet*, such as in Figure 1b and in the Stanford Math Circle logo in Figure 2.

a) 

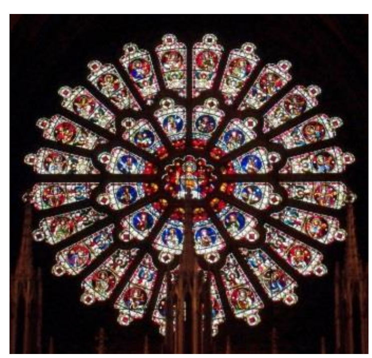
b) 

Figure 2: *a) The motif of the Stanford Math Circle, including a 5-5 kissing bracelet. b) The Rose Window of Durham Cathedral UK.*

The name “kissing bracelet” is based on:

a) a similarity with kissing circles,
b) their appearance like jewelry bracelets,
c) the use of “brace” to mean strengthen, a reference to the triangular properties of circle packing [11] and the use of triangles in engineering, well demonstrated in Figure 1.

When the number of circles in the inner ring is n and the number in the outer ring is kn, where k is an integer, then if each circle of the inner ring is tangent to two adjacent circles of the outer ring, we describe the bracelet as a *n-kn kissing bracelet* (Figure 3).

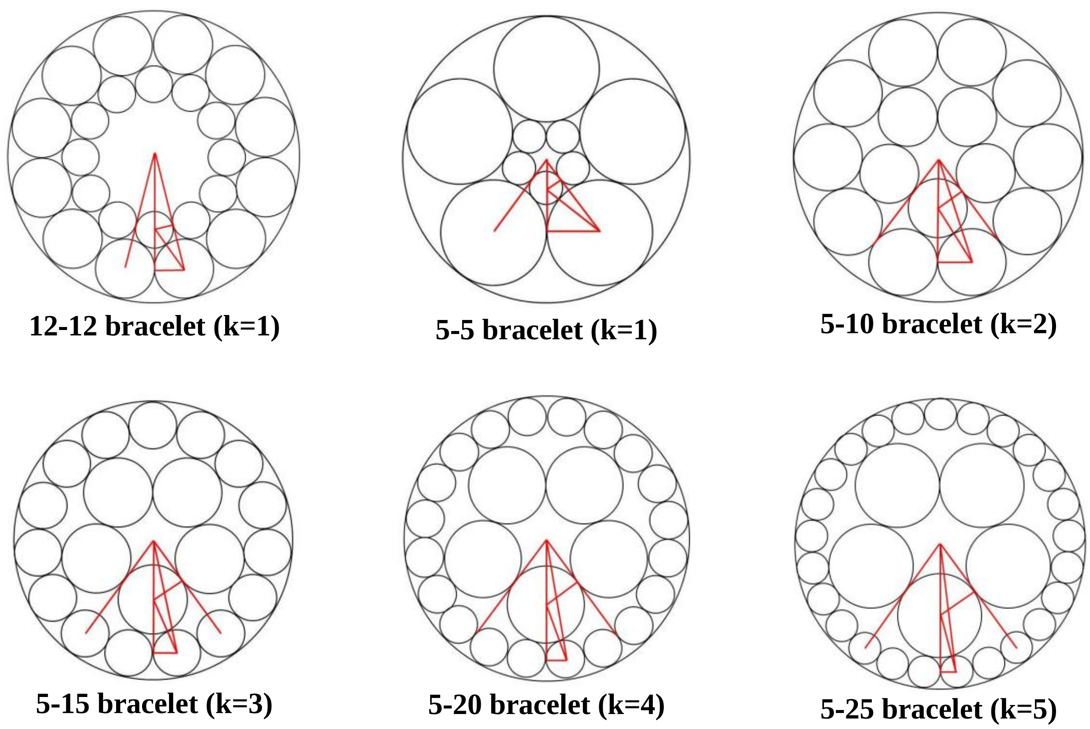


Figure 3: *Examples of n-n and n-kn kissing bracelets with trigonometry guidelines in red/bold.*

Circle Diameter Ratios. By applying simple trigonometry to an *n-n* kissing bracelet, such as the *12-12* bracelet in Figure 3, we can derive a relationship between the circles. Given two rings of n circles, the inner ring with circles of radius a and the outer ring with circles of radius b, for the closest fit the ratio $r=b/a$ of the diameters must be in accord with the following equation:

$$\frac{180}{n} = 90 - \left(sin^{-1}(\frac{1}{1+r}) + cos^{-1}(\frac{r}{1+r}) \right)$$

For multiple *n-kn* kissing bracelets, the ratio r of the radii of the circles in the outer ring compared to those of the circles in the inner ring, (Table 2), must be in accord with the quadratic equation:

$$x^2r^2 - (2xy + 2)r + (y^2 - 1) = 0 \quad \text{where} \quad x = 1/tan\left(\frac{180}{k \cdot n}\right) \quad \text{and} \quad y = 1/sin\left(\frac{180}{n}\right).$$

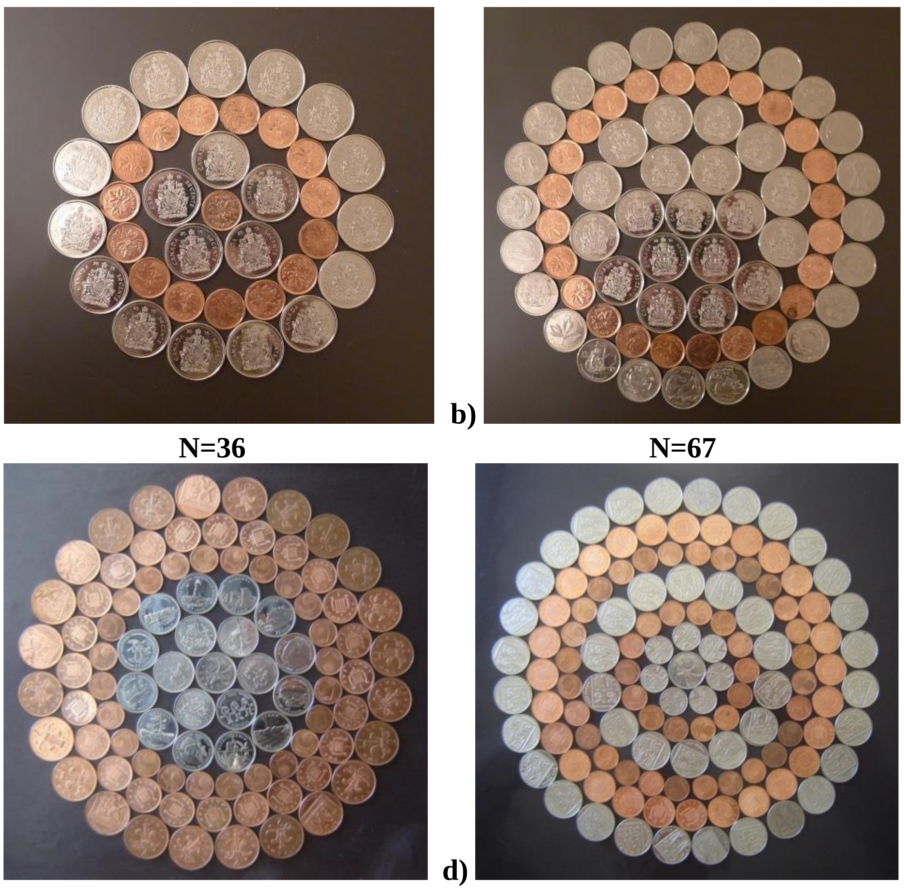
For *k=1,* similar values can be obtained using either of the above equations. For increasing values of *k* and *n*, approximate values of *r* have been derived using the quadratic formula to solve for the larger root (see Table 2). The reciprocal ratios *1/r=a/b* for *k=1* and *n=3*, *n=4* and *n=6* are in agreement the results of other researchers [12, 13, 14, 15].

n	3	4	5	6	7	12	14	15	24	28
r (k=1)	9.898	4.612	3.217	2.590	2.237	1.582	1.480	1.441	1.255	1.215
r (k=2)	1.943	1.391	1.142	1	0.909	0.714	0.679	0.666	0.599	0.584
r (k=3)	1.064	0.813	0.689	0.616	0.567	0.459	0.440	0.432	0.393	0.384
r (k=4)	0.730	0.573	0.493	0.444	0.412	0.338	0.324	0.319	0.292	0.286
r (k=5)	0.554	0.441	0.383	0.347	0.323	0.267	0.257	0.253	0.232	0.228

Table 2: *Kissing bracelets: values of r for various combinations of k and n*

Array Construction and Kissing Bracelets

Using the above information, a variety of circle arrays of *N* coins have been constructed (Figure 4). Given the coins used, readers can assess how closely the arrays match the theoretical best close fit.



a) **N=36** **b)** **N=67**

c) **N=91** **d)** **N=120**

Figure 4: *a) N=36 Pentagon centered rose with Canada 50 and 1 cent coins b) N=67 Hexagon centered rose with Canada 50, 25 and 1 cent coins c) N=91 Hexagon centered rose, with Canada 25 cent, Euro 1 cent, and UK 10 and 1p coins d) N=120 Heptagon centered rose with Canada 25 cent, Euro 1 cent and UK 10p, 5p and 1p coins*

The fitting of successive rings of circles where the number of rings doubles, for example from n=12 to n=24, is a common feature of rose windows as in Durham cathedral (Figure 2). Because of their similarity in appearance to these stained glass windows, we give multiple rings of successive kissing bracelets the name *kissing rose*.

Extending the Range: The maximum ratio of the diameters of any two Canadian coins is *r (max) =32.1/18.0 = 1.78* and of only the current coins is *r (max) = 28.0/18.0 = 1.56*. DVDs and jar lids can be used to increase the range and further test the mathematics involved [3]. CD/DVDs have the advantage that most are of a standard 120 mm diameter. The diameters of jar lids, of course vary greatly. Fortunately, the lids of the current 170g-200g jars of a brand of coffee, sold in Canada, UK and Argentina, and probably more widely, are of a common diameter (84mm). Together with the other coins mentioned, ratios available for study increase to *r (max) = 84/12.0 = 7.0*. With CD/DVDs *r (max) = 120/12.0 = 10.0*. Examples of close packing are shown in Figure 4. Interestingly, 5 CD/DVDs fit almost exactly around the described coffee jar lids [3].

a) b)

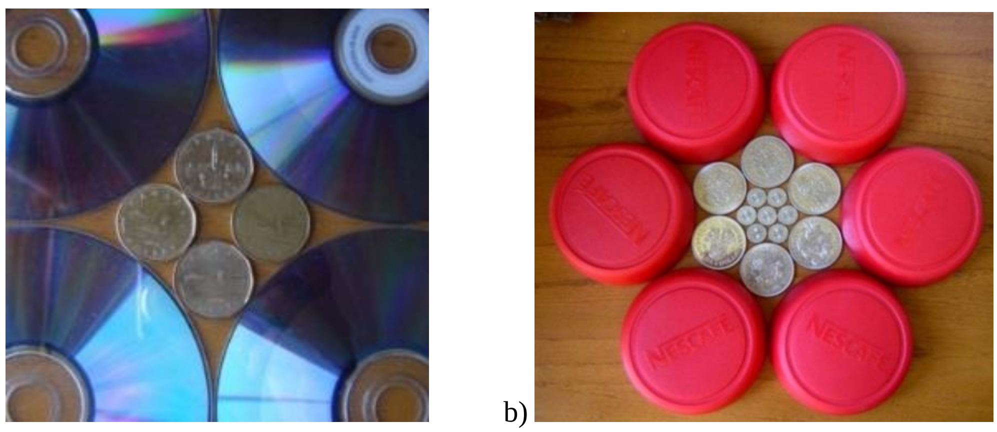
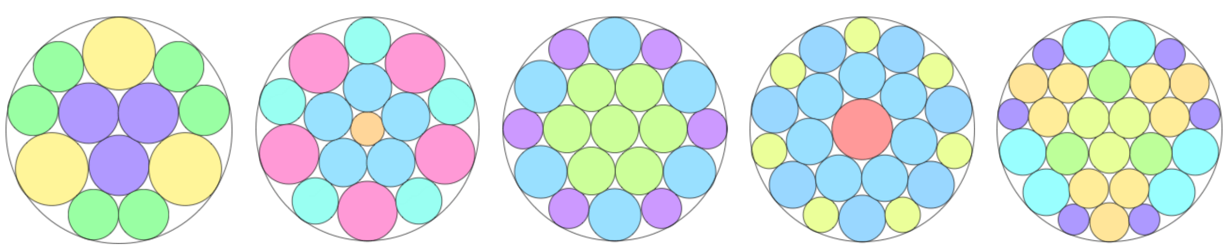


Figure 5: a*) 4-4 bracelet of Canada 1 dollar (post 1987, d= 26.5 mm) and CD b) 1-6-6-6 kissing rose containing central Uruguay 1 peso and successive 6-6 kissing bracelets of Uruguay 1 peso, Canada 1 dollar (pre 1987 d=32.1mm) coins, and coffee jar lids (d=84mm).*

Mixed Tertiary Arrays

The radii of the circles for the best fit in the close-packed circular arrays in Figure 6 have been determined and can be obtained through Erich Friedman's webpage [16]. Examples of coin arrays for N=12 and N=16 are shown in Figures 7 and 8 and 9.



N=12 **16** **19** **22** **27**

Figure 6: *Examples of symmetrical arrays of N discs with the outer ring containing discs of two or three different diameters*

Figure 7: *Reverse and obverse views of a hanging mixed tertiary array (N=12) of Canadian 1, 5 and 25 cent coins dating back to 1922. The blue background and coins are not in contact.*

The array N=12 in Figure 7 has been studied using discrete conformal mapping (see [17, 18]). The ratio of the three different radii for the closest fit inside a circle has been determined to be 1:1.20:1.44. The 1, 5 and 25 cent coins present here have radii in the ratio 1:1.11:1.25 so the coins do not nest in a circle. However, the closeness of the fit between coins is clear from the silhouette of the array photographed against a bright background (Figure 8).

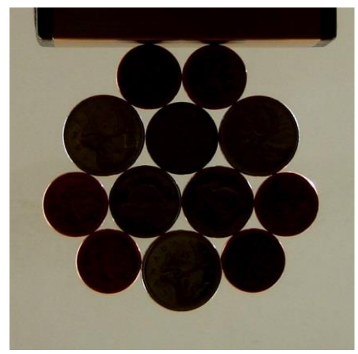


Figure 8: *Silhouette of N=12 array*

Figure 9: *Pentagonal based tertiary N=16 arrays containing Uruguay 1 cent, Argentina 5 cents, Canada 5 cents (standing left) and Canada 1 cent, Brazil 1 real, Canada 1 dollar (d=32.1).*

We can also construct a variety of aesthetically pleasing coin arrays which are not close packed, or, are not rings. Several such arrays were presented at Bridges 2015 [3]. Some analogues, built using Canadian coins, are shown in Figure 10.

Figure 10: *a) Standing array of current and old magnetic coins of Canada, based on the logo of the Royal Canadian Mint (shown above) b) Array consisting of repeating beetle-like units (patanons), circling a central decagonal array. Each patanon contains 13 larger and 10 smaller coins. This configuration can be extended indefinitely* [3].

To International Cooperation, Joy, Love and Peace through Art: A Final Conundrum and a Poem in Tribute to Frederick Soddy

Conundrum: Data from the Royal Canadian Mint indicate that since 1922, the total number of *circulating magnetic coins* produced was 2.7×10^{10}. Taking the average diameter of the coins as 20 mm for simplicity, a chain of these coins laid side by side would stretch some 500,000 km, more than sufficient to reach to the moon or wrap 10 times around the circumference of the earth. Put another way, this is a number sufficient enough to give each person in the world 3 magnetic coins, with some to spare.

What is the diameter of the smallest circle into which all the coins could be laid without overlapping?

On Waterloo Bridges, July 27, 2017

Has Canada anything to show more fair?
Dull would they be of soul who could not enjoy
Arrays so touching with Her Majesty.
The meeting now doth, like a garment, wear
The beauty of geometry; pair by pair,
Circles, patanons, rings and roses lie
Open to the fields, low and high;
All bright and glittering in the bracing air.
Never did math more beautifully steep
In its first splendour, drawing, note or bill;
Ne'er saw I, never felt, a force so deep!
The coins do lieth at their own sweet will:
Dear God! The very whiteboards seem asleep;
And all that magnetic art is kissing still!

Adapted from:
Composed upon Westminster Bridge, 3 September, 1802, by William Wordsworth.

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References

[1] I. Peterson. "Pennies in a Tray". *Science News*. 2004.

[2] R.L. Willson, P.A. Riley and D.J. Harris. *Investigating Magnetism. The Science and Art of Magnetic Coin Tricks, 4th Edition*. Brunel University London. 2013. ISBN: 1-872166-38-5, 1995. Available online in English and Spanish at: `http://www.capat.org/engIM4.htm` (as of April 5, 2017)

[3] R.L. Willson. "Magnetic Circle Packing in Creative Outreach and Refreshment". In K. Delp, C.S. Kaplan, D. McKenna and R. Sarhangi, editors, *Proceedings of Bridges 2015: Mathematics, Music, Art, Architecture, Culture*, pp. 275-282, Phoenix, Arizona, Tessellations Publishing, 2015. Available online at `http://archive.bridgesmathart.org/2015/bridges2015-275.pdf`

[4] Royal Mint of Canada. `http://www.mint.ca/store/mint/learn/canadian-circulation-1100028#.WOVVpk1-6Uk` (as of April 5, 2017)

[5] `http://www.magneticcoins.info/magneticcoindirectory.htm` (as of April 2, 2017).

[6] F. Soddy, "The Kiss Precise". *Nature*, 137, p. 1021. 1936.

[7] R. Williams. *The Geometrical Foundation of Natural Structure*: *A Source Book of Design*. pp. 35-39. Dover Publications. 1979.

[8] K. Stephenson. "Circle Packing: A Mathematical Tale". *Notices of the American Mathematical Society*, 50(11). 2003.

[9] K. Stephenson. "Introduction to Circle Packing". *The Theory of Discrete Analytic Functions*. Cambridge University Press, 2005.

[10] T.C. Inglis and C.S. Kaplan. "Circle patterns in Gothic Architecture". In R. Bosch, D. McKenna and R. Sarhangi, editors, *Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture*, pp. 133-140, Phoenix, Arizona, Tessellations Publishing, 2012.

[11] P.L. Bowers and K. Stephenson. "A regular pentagonal tiling of the plane". *Journal: Conform. Geom. Dyn*. 1 pp. 58-86, 1997.

[12] L. Fejes Toth. *Regular Figures*. The Macmillan Company, New York. 1964.

[13] O.U. Uche, F.H. Stillinger and S. Torquato, "Concerning Maximal Packing Arrangements of Binary Disk Mixtures". *Physica A*, 34, pp. 428-446. 2004.

[14] T. Kennedy. "Compact Packings of the Plane with Two Sizes of Discs". *Discrete and Computational Geometry*, 35(2), pp. 255–267. 2006.

[15] A. Heppes. "Some Densest Two-Size Disc Packings in the Plane". *Discrete and Computational Geometry*, 30(2), pp. 241–262. 2003.

[16] E. Friedman. `http://www2.stetson.edu/~efriedma/cirRcir/` (as of April 2, 2017).

[17] E.W. Weisstein. `http://mathworld.wolfram.com/CirclePacking.html` (as of April 2, 2017).

[18] V. Elser, In D.W. Cantrell, "Re: Maximizing the sum of radii of packed discs", 2011. `http://mathforum.org/kb/message.jspa?messageID=7500368` (as of April 2, 2017).