Similarity Drawn Freehand

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Abstract

A workshop about drawing similar polygons freehand. The goal is to generalize the similarity constructions for rectangles found in art and design books (which require no measuring or special tools), to similarity constructions that will work for *any* polygon and *any* similarity transformation. What is known about similarity if the ability to measure is taken away? How can angles and relative lengths be preserved in polygons that undergo rotations, reflections, translations, and dilatations? While primarily of interest those who draw or illustrate, even people without drawing experience can explore the ideas in this workshop and use some drawing aids to produce the same results.

Introduction

The drawings below are a set of five similarity constructions for rectangles that are commonly found in art and design books [1]. Each one of these similarity constructions was created using lines between points, parallel lines, and perpendicular lines. Notice that an artist with sufficient skill could draw all of these freehand, they wouldn't even need a straightedge. That's part of what makes these constructions special. There is something elegant and satisfying about doing something completely by hand. Drawing freehand is also integral to some kinds of art. To borrow the words of Arthur Guptill in the classic book *Rendering in Pen and Ink* [2] "... ruled lines are too straight and perfect to be interesting." In this spirit, all the figures in this paper were drawn freehand. Try drawing these similarity constructions yourself and think about why they work. If you don't have drawing experience, feel free to use graph paper, a straightedge, or other tools to make more accurate drawings.

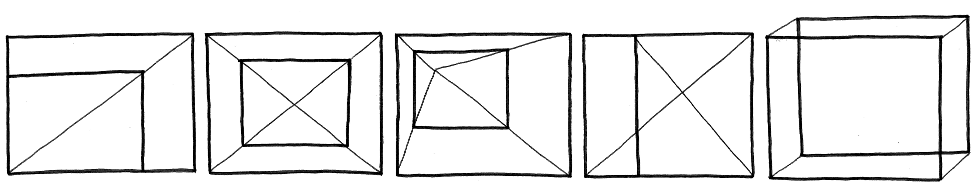

Figure 1 : *Similar rectangle constructions.*

Art and design books will point out the importance of the diagonal for similarity. Observe also that parallel lines and similar triangles play an important role. The second construction from the right is more complex than the others. Often referred to as the "reciprocal construction", it forms the basis for a system of proportion called Dynamic Symmetry that was created by artist and teacher Jay Hambidge [3].

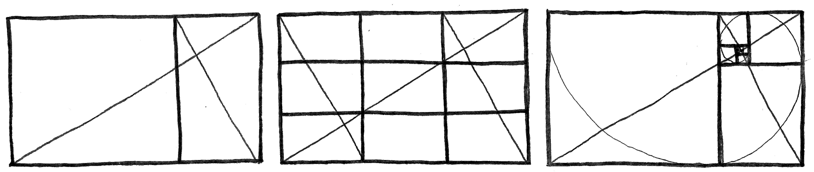

Figure 2 : *Examples of the reciprocal construction on a $\sqrt{3}$ rectangle.*

As versatile as it is beautiful, it can produce similar rectangles (shown above) with 90° rotations, scaling, and translation. The reciprocal construction was the first hint that more complex similarity transformations were possible. To get started, let's pin down some terminology. We'll assume that everything is restricted to a single two-dimensional surface, such as a piece of paper.

Polygon. *A geometric object consisting of a number of points (called vertices) and an equal number of line segments (called sides), namely a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points [4].*

Similarity transformation. *Any combination of translations, dilatations (or scaling), reflections, and rotations.*

Next, more specifics are needed on what similarity constructions mean in the context of freehand drawing. Similarity constructions are usually considered geometric constructions that result in a similar figure. Geometric constructions are drawings made with a compass and straightedge based on Euclid's Axioms. The first two axioms are reasonable to draw freehand:

I. A straight line segment can be drawn joining any two points.
II. Any straight line segment can be extended indefinitely.

Playfair's axiom (which is equivalent to Euclid's fifth axiom) states: "Given a plane containing a line and a point not on the line, there is a unique line parallel to the given line through the given point." This puts us in the realm of Affine geometry, which is sometimes called the study of parallelism [5]. Angles made with parallel lines are equal, and lengths can be compared on parallel lines using a parallelogram. However, to perform all similarity transformations, it's necessary to preserve angles and compare lengths on different lines even when the orientation is changed. Euclid's third axiom (that allows circles to be drawn with a compass) would work, but that axiom is more powerful than needed, and circles are significantly harder to draw than straight lines. Since similarity at its most basic simply needs to preserve the relationships that are already present, I propose the following alternative:

Reflection Axiom. Given two lines, there is an axis of symmetry between them, and a unique line can be drawn that is the reflection of a given line over the other.

Think of these three "similarity axioms" as a way to make sure the steps taken in the similarity construction are feasible to draw by hand as well as determine whether the resulting polygon is actually similar.

The reflection axiom is a strengthening of Playfair's axiom. A perpendicular line can be drawn by reflecting a triangle, and two perpendicular lines make a parallel line. They can be made to go through a certain point if it's used as the vertex of the triangle. When lines are reflected, their orientation changes, their angles relative to each other are preserved, and lengths can be compared on different lines using similar triangles. Furthermore, two reflections have the net result of a rotation! This axiom can be posed in a variety of ways, but the key idea is that one of the artist's basic tools is bilateral symmetry. A skilled artist can reliably double or bisect a length or an angle, and this idea of making two sides the same is what makes drawing parallel and perpendicular lines possible freehand. Bilateral symmetry is the easiest for us because it's part of our bodies and part of our minds as a sense of equality or balance.

More polygons and combining transformations

In the workshop, after we go over the similarity constructions for rectangles, we'll try to extend them to other polygons and try to combine transformations. The following are some examples. In the manner typical of a design book, a drawing of the similarity construction is provided and it falls to the reader to figure out how it was constructed and why the shape is indeed similar.

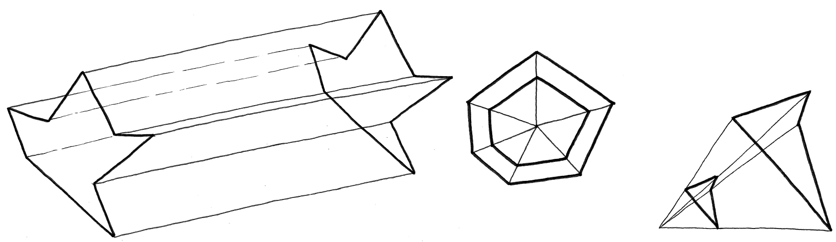

Figure 3 : *Translation and scaling.*

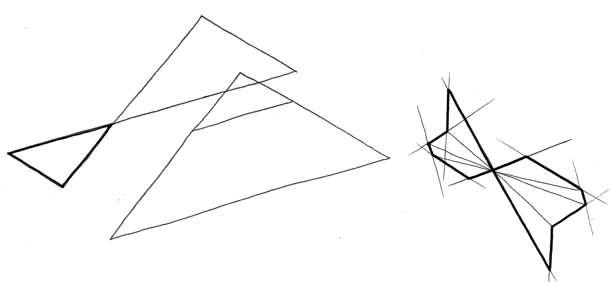

Figure 4 : $180°$ *rotations with translation and scaling.*

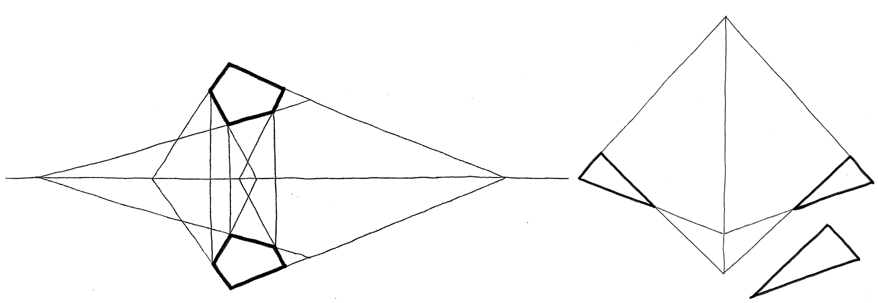

Figure 5 : *Reflections with translation and scaling.*

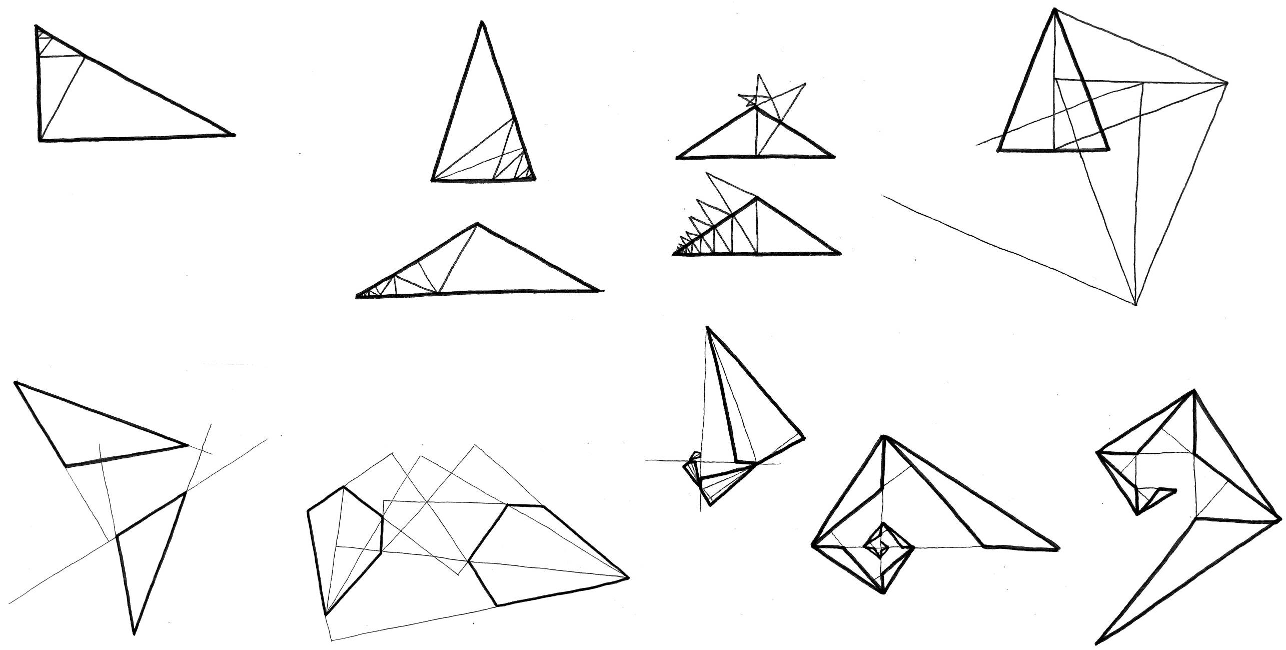

Figure 6 : 90° *rotations with translation and scaling.*

General Case

What remains is to combine all these possibilities into a single method and show that it works for all similarity transforms and all polygons. After workshop attendees have tried to make similar polygons themselves, they'll understand the scope of the problem better, and hopefully be able to understand the why and how. At first, they'll be given polygons drawn on graph paper to transform. This helps make sure the lines drawn have the correct slope, makes it clear that the resulting shape is similar, and avoids cases that are particularly difficult to draw. Afterwards, they can try polygons on blank paper or experiment with their choice of polygon and location, orientation, and size. Straightedges will be provided for those who wish to use them. In the workshop, the method will be presented as a step-by-step process whereas this paper will give a more mathematical treatment.

The first goal is to be able to perform any similarity transform on the simplest non-trivial polygon: a triangle. You'll see that these similarity constructions rely on reflected lines and parallel lines as set out in the similarity axioms. For convenience, a line that contains the side of a polygon will be called a "sideline".

Claim. *Given any triangle T, an arbitrary similar triangle T^* can be drawn with a freehand similarity construction.*

Justification. Let T be any triangle made up of sides q, r, s and vertices $q \cdot r$, $r \cdot s$, and $s \cdot q$. Draw an arbitrary line segment s^* in the plane that corresponds with s with endpoints $r \overset{*}{\cdot} s$ and $s \overset{*}{\cdot} q$.

If s and s^* are parallel, Draw a line parallel to q through $s \overset{*}{\cdot} q$. Draw a line parallel to r through $r \overset{*}{\cdot} s$. The resulting triangle T^* is similar to T because the angles are the same.

If s and s^* are not parallel, there exists an angle $\theta = \angle ss^*$. A triangle similar to T containing s^* could either have been rotated by θ, or since translation is involved, it could have an (improper) rotation by $180^\circ - \theta$. Which rotation is present depends on how the endpoints of s^* were labeled relative to the endpoints of s. A triangle similar to T containing s^* could have a reflection over the line that bisects θ, call this line the axis of symmetry.

If there is a reflection, reflect each sideline over the axis of symmetry. The reflection of sides q and r will be called q' and r' respectively. The angle between any two sidelines is preserved in reflection. Draw a line parallel to q' through $s \overset{*}{\cdot} q$. Draw a line parallel to r' through $r \overset{*}{\cdot} s$. The resulting triangle T^* is similar to T because the angles are the same.

If there is a rotation, we'll use the fact that performing two reflections has the net result of a rotation. First reflect the sidelines containing p and q over the the sideline containing s. This delineates a triangle of the same size that is similar because the angles are the same. Reflect this triangle over the axis of symmetry as before. This results in a triangle T^* similar to T containing s^*.

Since s^* can be any size and in any location, any translation, scaling, and reflection is possible. Any rotation can be achieved because rotation can be clockwise or counter-clockwise based on the relative position of s and s^*. Therefore, the choice of s^* and the orientation describes an arbitrary similarity transform. All the lines and points can be drawn with a freehand similarity construction.

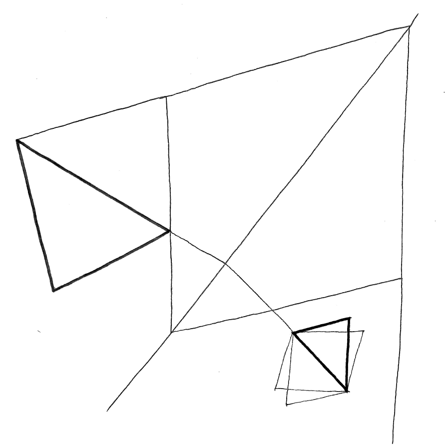

Figure 7 : *Constructing a similar triangle.*

Polygons with the same angles are not necessarily similar. To create similar polygons we'll use a special kind of triangulation to maintain the ratio of the sides and help reconstruct the shape using the previous result.

Claim. *Given any polygon P, an arbitrary similar polygon P^* can be drawn with a freehand similarity construction.*

Justification. Let P be any polygon. Pick a side s of P. Draw an arbitrary line segment s^* in the plane that corresponds with s. Choose a point that is not on any sidelines of P as a center. Draw lines between the center and each vertex of P. This means that the endpoints of each side of P are not collinear with the center and a triangle is formed for each side. These triangles all share a vertex at the center and adjacent sides of the polygon share an interior side of a triangle.

Reconstruct the triangle containing s at s^* using the previous claim. This constructs the center for P^* and the side for an adjacent triangle that can, in turn, be reconstructed. Since we are on a polygon, there is a connected path of triangles that share sides. Each triangle reconstructed will create a new side of P^*.

The triangles are similar because the angles are the same, so the angles of the polygon are the same as well because we used the same pattern of adjacent triangles. The ratio of the sides is preserved in the triangles, and since they all share sides, the ratio of the sides of the polygon are also preserved as well. Therefore, P^* is similar to P. As before, the choice of s^* and the orientation describes an arbitrary similarity transform and all the lines and points can be drawn with a freehand similarity construction.

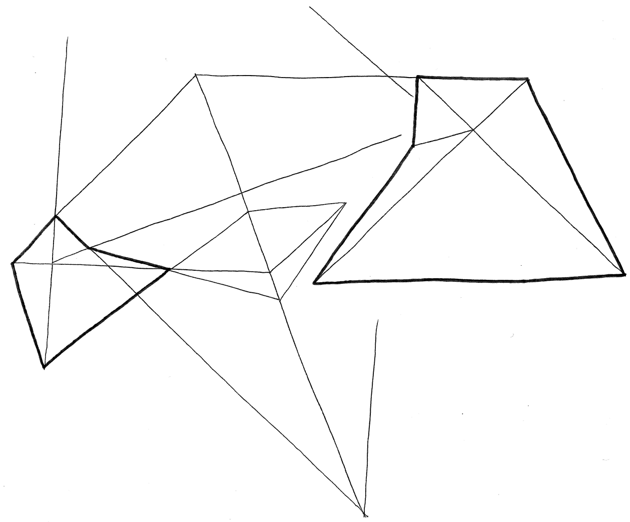

Figure 8 : *Constructing a similar polygon.*

Experiment with these ideas and find the method that suits you best. For example, try using vertices or the intersection of two diagonals for the center of the triangulation, try other lines of reflection to create rotations, or even an angle template to rotate all the sides. Consider this method a starting place, it can be improved upon with more thought and practice.

For further study, play with the similarity axioms or explore the topic in the setting of projective geometry: projectivities, complete quadrilaterals, projective collineations, perspective drawing. Could Desargues' Theorem be modified to allow for the changes of orientation in similarity transformations? What triangulation of the polygon would be most efficient? What characteristics could make polygons easier to transform? What other mathematical results about similarity can be applied to art?

These freehand similarity constructions preserve the relationships present using little more than parallel lines and reflections, and the artist has the freedom to choose any location, size, and orientation by eye to suit the composition of the artwork. The versatility, simplicity, and symmetry in the freehand similarity constructions make them, in my opinion, aesthetically pleasing in both a visual and mathematical way. Hopefully this also makes them a useful tool for artists seeking to use similarity in their work.

References

[1] K. Elam, *Geometry of Design: Studies in Proportion and Composition* (2001) Princeton Architectural Press.

[2] A. Guptill, *Rendering in Pen and Ink* (1976) Watson-Guptill Publications.

[3] J. Hambidge, *The Elements of Dynamic Symmetry* (1926) Brentano's, Inc.

[4] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited* (1967) Mathematical Association of America.

[5] E. Lord, *Symmetry and Pattern in Projective Geometry* (2013) Springer.