The Fourth Dimension in Mathematics and Art

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Abstract

The fourth dimension is a complex concept that deals with abstract reasoning, our sense of perception, and our imagination. This paper gives a brief overview of the background that leads to the study of the fourth dimension and focuses on the specifics of mathematics and visual imaging to illustrate the challenges and benefit of interdisciplinary collaboration to create a sound outcome.

Introduction

There is something I 'know', which is that spatial dimensions beyond the Big 3 exist. I can even construct a tesseract or hypercube out of cardboard.
David Foster Wallace [1].

Scientists and philosophers have studied the concept of dimension since the beginning of time. Greek philosopher Aristotle and later mathematician Ptolemy argued that no more than three spatial dimensions were possible. Plato in the *Allegory of the Cave* tells us that concept and perception are two distinct things Between 1826 and 1829 mathematicians Nicolai Lobachevski and János Bolyai constructed a self-consistent system of geometry in hyperbolic space that permits an infinite number of straight lines to pass through a point parallel to a given line [3]. Their theory for the first time in two thousand years questioned the fundamental principle of Euclidian geometry. It lead German mathematician George Riemann to develop a new kind of differential geometry of space with any numbers or dimensions and curvature.

The Fourth Dimension in the Visual Arts

French mathematician Maurice Princet is credited with opening the door to the understanding of Riemann and Poincaré theories to people like Picasso, Duchamp, Gleize and Metzinger in series of informal lectures and talks. Dr. Linda Henderson in an essay on the fourth dimension emphasized that the cubist perspective helped open the understanding of a material quantifiable fourth element that later Dali symbolically enshrined in the 1954 Crucifixion (Corpus Hypercubus). It depicts on a flat surface what the unfolded three-dimensional net of a four-dimensional hypercube could look like; it is now exhibited at the Metropolitan Museum of Art in New York City where it has drawn crowds ever since [4].

Today, the use of spatial representation to visualize a fourth dimension can be found in architecture as well. The massive skyscraper "*La Grande Arche"* was designed by architects Von Spreckelsen and Reitzelin to look like a hypercube projected on a three-dimensional world [5]. Located in the outskirt of Paris, it is visited by more than 5 million tourists every year (Fig. 1).

Figure 1: *La Grande arche. Paris-La Defense, France.*

Computer Aided Visualization of a Fourth Dimension

Mathematician and topologist Jeffrey Weeks, Mc. Arthur recipient and 2007 AMS Conant prize's recipient for his research on the Poincare dodecahedral space, developed a series of mathematical software for geometry students to help them visualize mathematical constructs. One of the programs, 4D Draw [6], convincingly brings us closer to the perception of a fourth dimension on a two-dimensional surface. Points and lines on the stage can be tilted, turned upside down inverted and rotated up to 360º to allow the user to build meshes depicting 2D, 3D and 4D surfaces.

What starts as deceptively simple vector based geometry on a flat workspace becomes more and more complex as points and lines are added to the original shape. The program's algorithm monitors every new additional element introduced on the stage, accepts it, guides us to reposition it if needed, or refuses if it is improperly placed. In addition, points can be re-colored to help the viewer understand and differentiate each point positioning and hierarchy in this virtual space (Fig. 2).

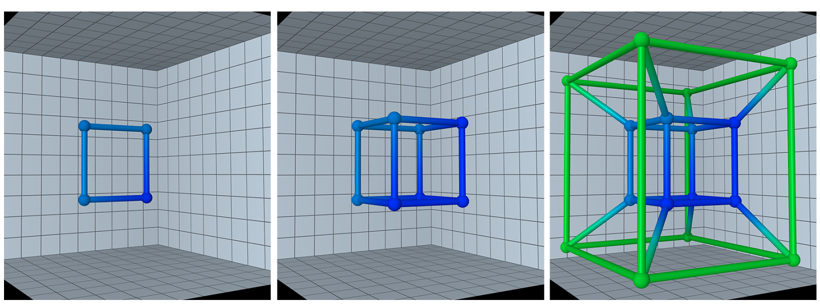


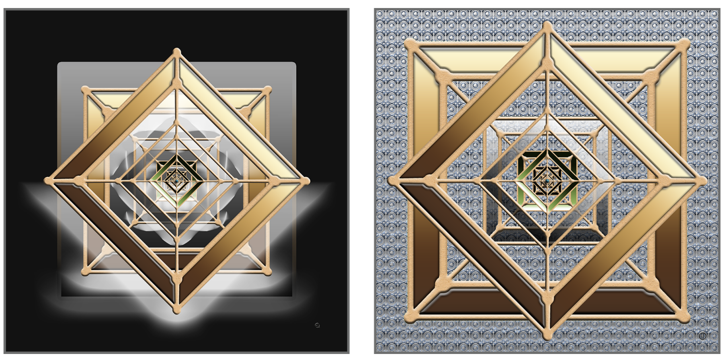
Figure 2: *Left to right - from a 2D outline, to a 3D model, to a 4D projection.*

Following are a few examples of unconventional results one can build from Dr. Weeks's 4D Draw program. In the first example, I used the program to create a basic four-dimensional figure called a tesseract, or hypercube. I exported the resulting image in various two and three-dimensional modeling programs to extract and emphasize elements that seemed relevant to the final visual statement (Fig. 3).

I first migrated the 4D Draw document in a two-dimensional vector based graphic editor, in this case Adobe Illustrator. Vector graphics are made of paths defined by a start and end point, along with other points, curves and angles. The lines, colors, curves, and all other geometrical attributes are stored in the form of mathematical formulae. Media Arts professors Reas and Fry sum up the flexibility of the formula in a programming handbook for visual designers and artists [7]: "Every time an image is constructed, those equations create an image scalable to any size and detail, regardless of the screen dimension and resolution with no noticeable degradation of quality even if the image is made a thousand-time larger or smaller."

In a second step, I migrated the document back into a bitmap environment to increase the image contours and redefine the volume and form of the outcome. In a bitmap image, rows and columns of dots create a picture. Bit-mapped graphics, or raster graphics, are the choice format to create visualizations that look like a photograph of an existing object or a photo-realistic composition.

And last, I imported the image in a three-dimensional programming environment to accentuate various elements of the emerging form. 3D Autodesk Maya software is built around the concept of modular nodes or vertices. Networks of interconnecting nodes depend on each other's information. Its powerful algorithm and the sophisticated light tools one can use on the program's stage help build additional depth and definition otherwise difficult to obtain in a pixel based, two-dimensional environment. The following illustration (Fig. 3) represents two variations of the resulting design in a different context to emphasize the strength and complexity of the original shape. The strong black background on the left guides the eye to focus on the center of the composition and privileges depth and perspective; the textured light background on the right side invites a more wholesome appreciation of the object, highlight the contouring of the lines and what the representation of a 4D concept could look like on a two-dimensional surface.



Figures 3: *Examples of the front view of a hypercube reworked on various 2d graphic editors.*

Another intriguing four- dimensional figure is the octaplex. Also called an icositetrachoron, this construct is a finite regular four-dimensional polytope composed of 24 octahedra, with 3 to an edge. The 24-cell has 24 vertices and 96 edges and is one of the six regular polychora. [8]. Mathematician P. Bourke stated that it has no equivalent in other dimensions [9]. It is another good example of how a mathematical demonstration can inspire an artistic vision.

Representing 96 faces on a simple two dimensional flat surface is a technical challenge if one is to both respect the intent of the demonstration and meet the expectation of a viewer facing a flat screen or a canvas on a wall. To that end, I anchored the core definition of the design from a top view perspective, an orthographic projection with the center as the unique vanishing point from which to build the illusion of distance. The octaplex, along with numerous other mathematical 4-dimensional figures and diagrams, features prominently in geometer and mathematician H. S. M. Coxeter extended volume on groups and polytopes [10]. I used one of P. Bourke's simplified versions of the Coxeter original blueprint to extract the essence of the object and retain the complexity of its definition. It allowed me redirect my attention toward the creative aspect of the process and develop an aesthetically coherent statement. I rebuilt the blueprint in 4D Draw and developed two variations of the projection into a single image following the three steps process I used previously for the hypercube. The result is now being studied in a gemology laboratory to evaluate the possibility of converting it into a three-dimensional object that reflects the aesthetic representation of a four-dimensional concept (Fig. 4).

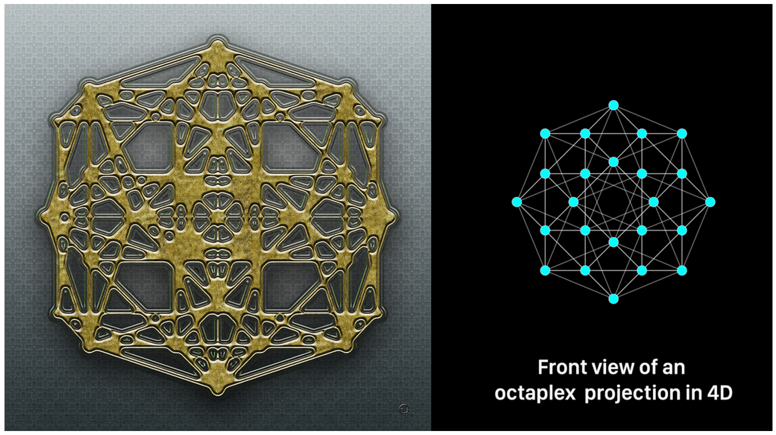


Figure 4: *Front view of a 24 cell-octaplex.*

Conclusion

Artists and architects have been trained since antiquity to project three-dimensional perspective on two-dimensional drawing. Today, computer based technologies such as ray tracers and CAD software allow mathematicians to convert complex abstract concepts into meaningful visualizations to reach a much larger audience. Oculus Rift, Google Cardboard and other VR technologies are already attracting and challenging engineers, architects and artists to outdo themselves and each other in this new environment.

Because of its simplicity, 4D Draw stands as an excellent educational tool to train newer generations in this exciting journey of discovery and help push further the frontier of science and knowledge-based visualization. Mathematics is the language with which tangible discoveries are made. Visual art is the tool that helps us understand and communicate better these new and challenging concepts. Both are grounded in physical reality. Future collaborations between the two disciplines are bound to benefit each other and help the public understand better our visible and projected environment.

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