Polygon Spirals

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Abstract

Logarithmic spirals may be classically constructed with a chain of similar triangles that share the same center of similitude. We extend this construction to chains of n-gons with centers on a logarithmic spiral with turning angle π/n, and scale factors with interesting properties. Finally, polygon spirals of this kind are used to produce a variety of artistic images.

Introduction and Motivation

The investigation of polygon spirals began by studying curves that arise when regular polygons with an odd number of sides are strung together. When polygons are strung together into a band by gluing them together along their sides, then the choice of what subsequent edge pairs are being used will define a turning angle introduced at each joint. In this paper we focus on bands made with minimal turning angles and with a consistent turning direction. Each odd n-gon defines its own turning angle, π/n. Moreover, by introducing a constant scale factor that modifies each subsequent polygon, a large variety of logarithmic spirals can be generated.

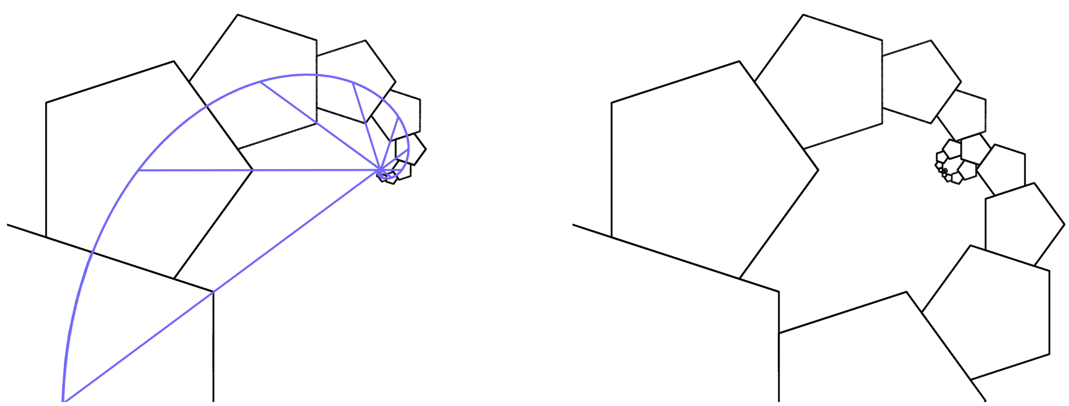

Figure 1 : *Logarithmic construction on pentagons (left) and spirals in phase (right).*

My original inspiration came from observing spirals of opposite handedness emerging from adjacent faces of the same polygon. The natural question that arose was which ratio to pick so that the two polygon chains would fall in phase, as in Figure 1 (right). In other words, I needed to find the ratio such that every crossing point of the two logarithmic spirals coincided with the center of a polygon along each band. In this case, the two bands would share a polygon every period. I found this to be achieved when the golden ratio was applied to pentagons, which spurred a determination of the analogous ratio for generalized n-gons that corresponds to the sharing of every n^{th} polygon. The construction hinges on similar triangles whose vertices are the center of a polygon, the center of one of that polygon's children, and the center of similitude.

Referring to Figure 2 (illustrated in the case $n = 5$), $\alpha = \pi/n$ and $\beta = 3\pi/n$, so that $\gamma = \pi/n$. Now the triangle with angles labelled α, β, and γ is isosceles because $\alpha = \gamma$. Moreover, the ratio between the longer and shorter sides is the same for all triangles since the spiral is logarithmic. If the shorter sides are of unit length, half of the base is $\cos(\pi/n)$, making the base, and thus the ratio, equal to $2\cos(\pi/n)$.

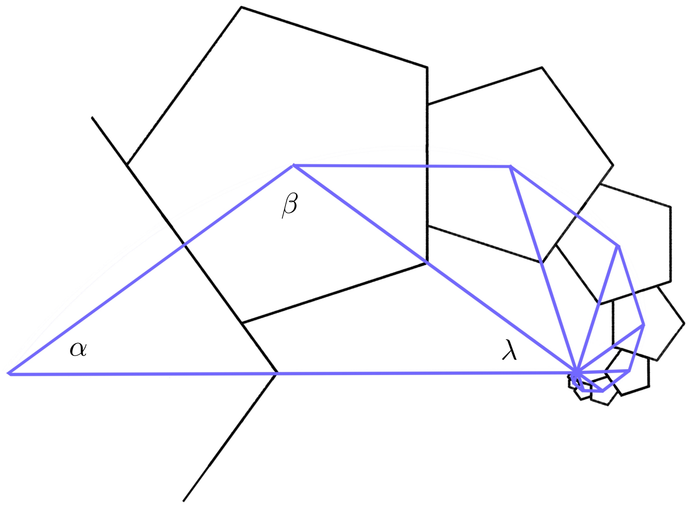

Figure 2 : *Logarithmic similar triangles.*

Once the ratio has been found, spirals can be nested by applying a rotation of π/n. Completing this process yields n nested n-gon spirals. With the ratio found above, it is not hard to show that the equation for one of the logarithmic spirals with this ratio, passing through the centers of the n-gons, has the equation:

$$r = (2\cos(\pi/n))^{n\theta/\pi}.$$

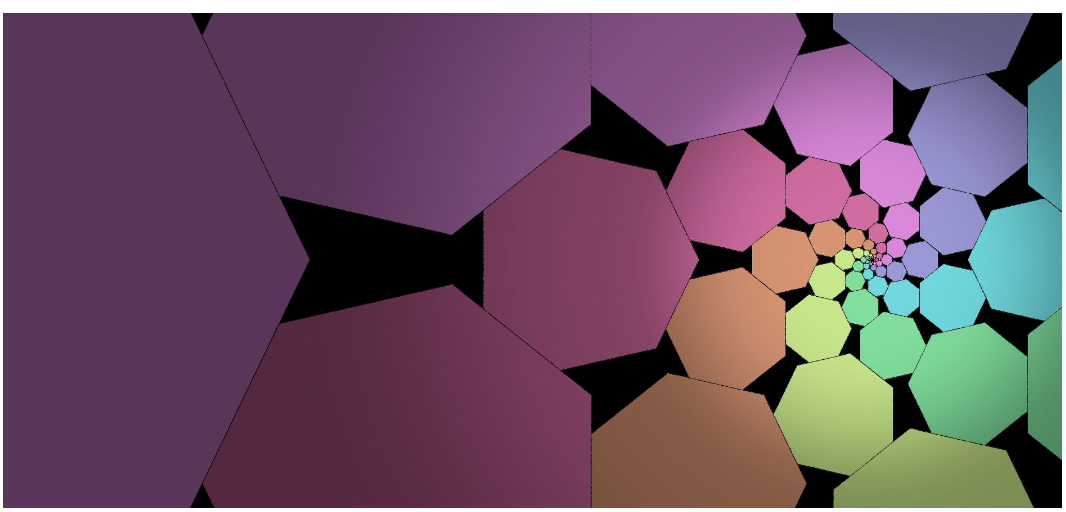

Figure 3 : *Heptagon spiral.*

Artwork

Subsets of the space including all possible spirals exiting from every face at every iteration are generated using a random algorithm. This algorithm assigns a probability that a spiral is generated from each face of the n-gon, and this probability is randomly altered and then inherited by its child. The colors are also inherited and randomly altered every generation. In Figure 4, the detailed texture is actually built of many small pentagons at a deep iteration. Off to the right can be seen a randomly generated pair of pentagonal arms falling into phase, the structure which inspired the particular subset of the space to use.

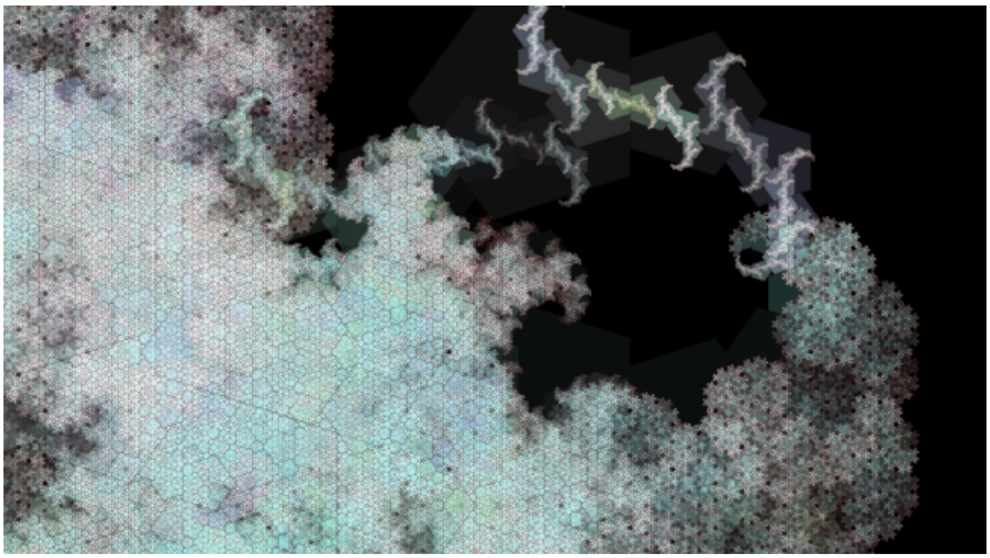

Figure 4 : *Pentagon arms.*

Figure 5 (left) is a tiling of pentagons that features nested rings of pentagons with the property that any two adjacent pentagons differ in size by a ratio of the golden ratio. To the right is an overlay of nonagon spirals with ratios between 0 and 1. This image captures the vast breadth of possible spirals based on a given n-gon, and the fascinating way that they interact.

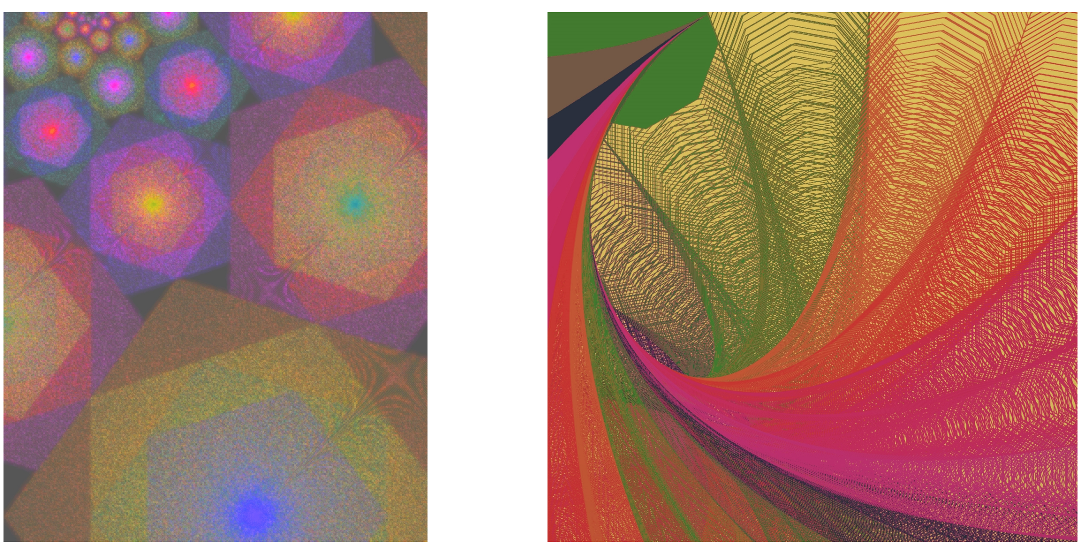

Figure 5 : *Nested pentagon tiling (left) and nonagon spiral overlay (right).*

Figure 6 was randomly generated by the same algorithm which produced Figure 4, however with nonagons rather than pentagons. This picture illustrates the infinite detail of a fractal set based on interacting nonagon spirals.

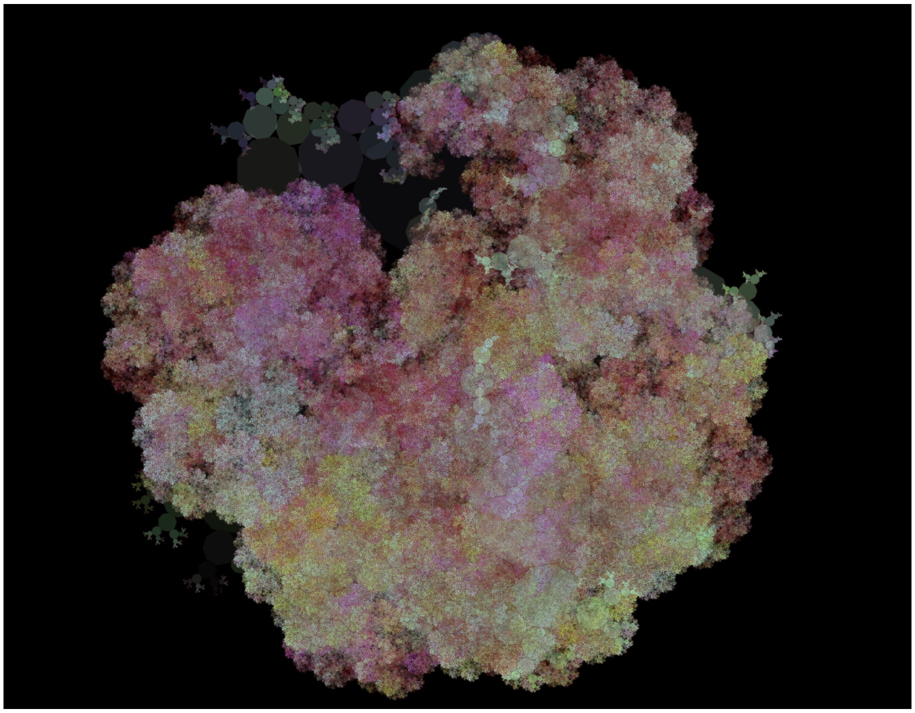

Figure 6 : *Nonagon bush.*

Concluding Remarks

Discovering the ratio which allowed polygon spirals to nest resulted in the art work shown in Figure 5. These images of nested polygon spirals inspired an exploration that led to the more advanced randomly generated polygon trees in Figures 4 and 6. I hope these images inspire continued interest in geometric beauty and recursion.

Acknowledgments

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References

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