From Klein Bottles to Modular Super-Bottles

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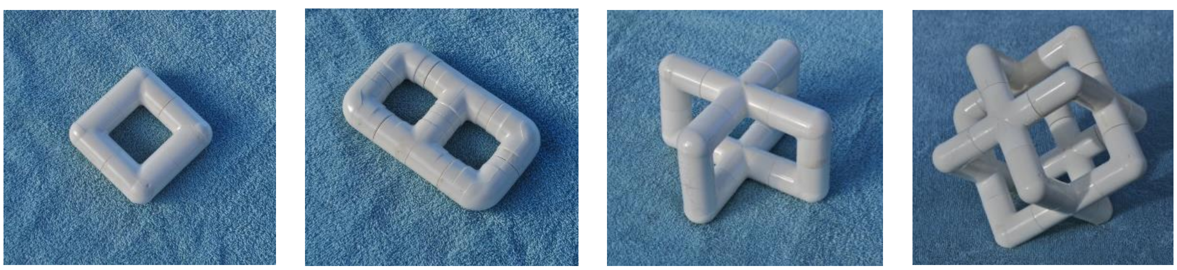
Abstract

A modular approach is presented for the construction of 2-manifolds of higher genus, corresponding to the connected sum of multiple Klein bottles. The geometry of the classical Klein bottle is employed to construct abstract geometrical sculptures with some aesthetic qualities. Depending on how the modular components are combined to form these "Super-Bottles" the resulting surfaces may be single-sided or double-sided.

1. Double-Sided, Orientable Handle-Bodies

It is easy to make models of mathematical tori or handle-bodies of higher genus from parts that one can find in any hardware store. Four elbow pieces of some PVC or copper pipe system connected into a loop will form a model of a torus of genus 1 (Fig.1a). By adding a pair of branching parts, a 2-hole torus (genus 2) can be constructed (Fig.1b). By introducing additional branching pieces, more handles can be formed and orientable handle-bodies of arbitrary genus can be obtained (Fig.1c,d).



(a) (b) (c) (d)

Figure 1: *Orientable handle-bodies made from PVC pipe components: (a) simple torus of genus 1, (b) 2-hole torus of genus 2, (c) handle-body of genus 3, (d) handle-body of genus 7.*

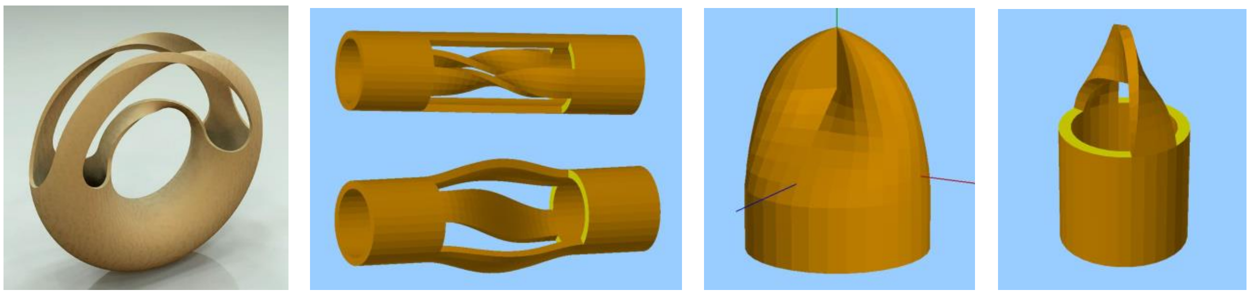
(a) (b) (c)

Figure 2: *Handle-bodies of genus zero: (a) with 2, and (b) with 3 punctures, (c) with zero punctures.*

In these models we assume that the pipe elements represent mathematical surfaces of zero thickness. Thus a simple pipe segment (Fig.2a) is used to represent a surface of genus zero with two punctures, even though this thick physical object really has the topology of torus. As long as the combination of such pipe elements maintains the connectivity of a tree, i.e., a graph with no loops, the surface will always be of genus zero. Open ends simply represent punctures in this surface (Fig.2b). When all the arm-stubs are properly capped, the surface has zero punctures (Fig.2c).

2. Single-Sided, Non-Orientable Handle-Bodies

To make models of single-sided, i.e., non-orientable surfaces, we need some parts that cannot readily be found in a hardware store. But all that is needed is a single component that somehow connects the two sides of the surface to one another. This could be done by introducing a pipe segment that is split longitudinally into a set of ribbons that twist through 180° before they join again into a complete pipe, as illustrated in Roelofs' "Moebiustorus" [4] (Fig.3a); but this element only serves its purpose, if it appears in a loop in the pipe system. On the other hand, if only some of the ribbons are twisted and others are left un-twisted, or if one ribbon travels across the center of the pipe, then the pipe segment itself becomes single-sided (Fig.3b); any pipe-system into which it is inserted becomes a non-orientable 2-manifold.



(a) (b) (c) (d)

Figure 3: *Turning a pipe assembly into a single-sided 2-manifold: (a) Roelofs' "Moebiustorus" [4], (b) single-sided pipe segments, (c) cross-cap, (d) simple Möbius pipe-end-cap.*

Alternatively, one can construct pipe-end-caps that are single-sided. Such an end-cap could take the form of a cross cap (Fig.3c) or of a Boy cap [7], or it could have just a single twisted strap (Fig.3d) that makes it equivalent to a Möbius band with a puncture. The introduction of just one such single-sided component will render any pipe system single-sided and non-orientable. Adding multiple such elements will keep the surface single-sided, but it will increase its genus.

The described components can yield the desired topological results, but the aesthetic appeal of these constructions is rather limited; they lack the simple elegance of the classical Klein bottle (Fig.4a). Thus the goal of this work is to look for other types of modular components that could produce non-orientable 2-manifolds of higher genus, and which would make use of the geometry of the classical Klein-bottle mouth ("KBM"). Cliff Stoll [12] connected two such elements together (Fig.4b) – however, in this case the result is a double-sided surface of genus 1, i.e., just a torus with some self-intersections. Forming a loop from an odd number of Klein bottles [6] results in a single-sided surface; but its genus is still only 2. In order to make surfaces of higher genus, some kind of branching module is needed. Inspired by project LEGO-Knots [8], [9], I set out to find a modular component that could yield attractive surfaces of higher genus – single-sided (non-orientable) as well as double-sided (orientable).

(a) (b) (c) (d)

Figure 4: *Geometries using the classical Klein-bottle mouth ("KBM"): (a) "Inverted Sock" Klein bottle, (b) Cliff Stoll's "Double Klein Bottle" [12], (c) a 3-way junction incorporating a KBM; (d) two such elements connected into a 2-sided surface of genus 2, equivalent to a 2-hole torus.*

3. A First 3-Way-Branching Klein-Bottle Mouths (KBM) Component

Figure 4c shows a first component that smoothly integrates the geometry of the classical KBM with the required branching function. Two such pieces can be assembled in three different ways. In one case, when the two thinner arms emerging from the center of the tori are connected to one another (Fig.4d), the result is a double-sided surface of genus 2, equivalent to a 2-hole torus with two punctures; this configuration has the same topology as Figure 1b with two holes inserted. In the other two cases, the result is a single-sided surface of genus 4, corresponding to the connected sum of two Klein bottles. The resulting sculpture (Fig.5a) was exhibited at the Exhibition of Mathematical Art at the Joint Mathematics Meeting 2016 in Seattle [3].


(a)


(b)

Figure 5: *Super-Bottle sculptures exhibited at the JMM 2016 [3]: (a) genus 4, (b) genus 10.*

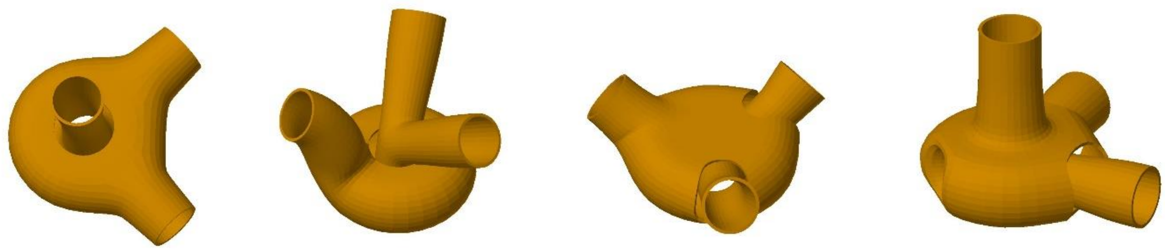
4. Modular Cube-Corner Components

The result shown in Figure 5a is aesthetically superior to any assembly made from PVC pipe segments. The challenge now is to find a small set of truly modular components offering similar aesthetic appeal, and which can be connected in various ways to make surfaces of different genus. For artistic reasons and to achieve a uniformity of style, I focused on solutions that integrate the branching function with the needed surface eversion and achieve this with the use of the classical Klein bottle mouth (“KBM”).

There were some other high-level decisions I had to make early on, driven by aesthetic concerns for the overall assembly. I wanted the overall tubular structure to follow the edge-frame of a regular or semi-regular polyhedron. To enhance modularity and to minimize complexity and the number of parts that I would have to fabricate, I first limited myself to cubic graphs; this implies that all the nodes would be realized by 3-way junctions. This choice singles out three of the regular polyhedra among the Platonic solids and also allows the assembly of regular *n*-sided prisms or of the truncated versions of all Platonic solids. Of course, all of these polyhedra have different angles between the edges emerging from the cubic vertices; but to make a truly modular component, a specific branching angle has to be chosen. I chose the cube corner as a versatile compromise. Eight such elements can be connected directly to form a cube frame – which yields a sculpture of moderate complexity (Fig.5b); this sculpture was also exhibited at JMM 2016 [3]. Furthermore, with the use of a small set of curved connector pieces, several of the other cubic graphs mentioned above can be realized as well.

Since I wanted a fair number of these components to play with, ease of fabrication and minimizing the amount of material used (thereby also minimizing cost and build-time) was foremost on my mind [10]. Because I wanted to use some kind of KBM geometry, the parts will exhibit much concavity and hollow regions. Layered manufacturing is a good technique to fabricate such geometries; in particular, Fused Deposition Modeling (FDM) with a soluble support structure [2] or Selective Laser Sintering (SLS) [5] can provide robust parts – often in brilliant colors – at relatively low costs.

Next, I wanted to combine the needed branching function with the geometry of the KBM in the most integrated, aesthetically pleasing manner. In the classical Klein bottle mouth there is an inner, thinner tubular segment turning "inside-out" into a thicker, outer tubular part. The branching can be introduced either into the thicker outer part or into the thinner inner arm. I studied a few different geometries exploiting these various possibilities, yet they should all clearly belong to the same "family". I liked the toroidal geometry of the classical KBM and I wanted to incorporate this into all the components as a common style element. Figure 6 shows the arrangements that I chose for further development and physical realization. In the first one (Fig.6a), the branching occurs in the thicker outer portion of the torus. In the other three solutions the thinner arm going to the inner part of the torus incurs the branching; in Figure 6b this happens before this arm enters the torus. In Figures 6c and 6d this happens inside the toroidal body.



(a) (b) (c) (d)

Figure 6: *Four different ways of making a 3-way KBM-junction.*

Klein bottles cannot be embedded in 3-space without self-intersections. I eliminated all self-intersections by cutting appropriate punctures into the surface, wherever an "inner" branch of a Klein bottle mouth (KBM) had to pass through the wall of a thicker branch or was exiting from within the main toroidal body. This avoided the creation of closed-off internal pockets, where the scaffolding material used in the layered manufacturing process can become trapped.

5. Modular Sculptures Based on Cubic Graphs

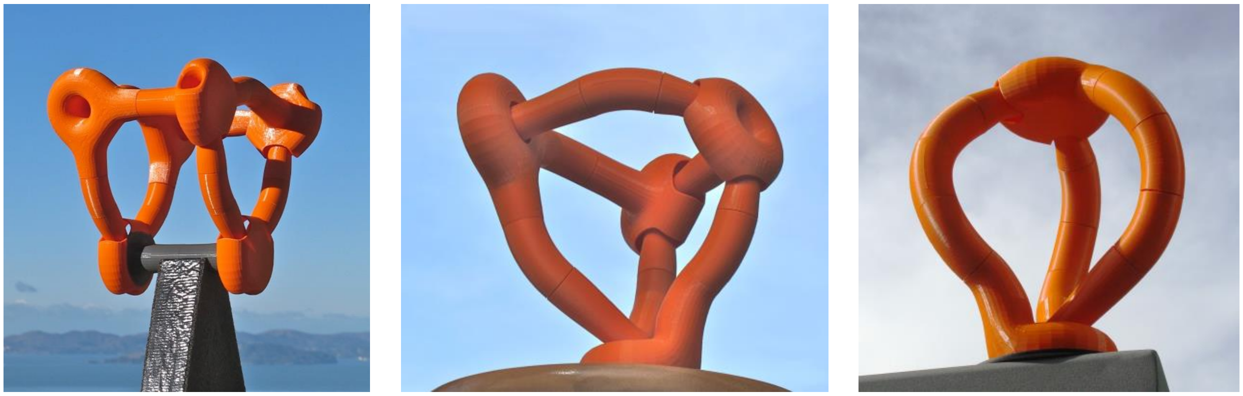
I use two of each of these modules to build a sculpture based on a polyhedral cube frame. Module 6(b) can readily serve as a stand for this sculpture, balancing it around one of its space diagonals (Fig.5b). Of course, there are many different possibilities to place the different modules at different corners and to rotate them in place through three different orientations. In most cases, the resulting surface will be single-sided; but in a few more symmetrical arrangements it will result in a two-sided surface of genus 5.

By introducing curved connectors bending through 30°, three cube-corner components can be readily joined into a 3-ring, and two such rings can be joined directly to one another to yield a 3-sided prism structure (Fig.7a). This yields a Super-Bottle of genus 8/σ, where σ denotes the sidedness of the surface.

Four cube-corner modules can also be used to make a tetrahedral frame. Since the angles between the arms of the cube-corner component are more obtuse than the 60° angles required in the tetrahedron, we need to introduce curved connector pieces that bend through 38.96° to form the curved edges of a tetrahedral frame (Fig.7b). This yields a Super-Bottle of genus 6/σ.

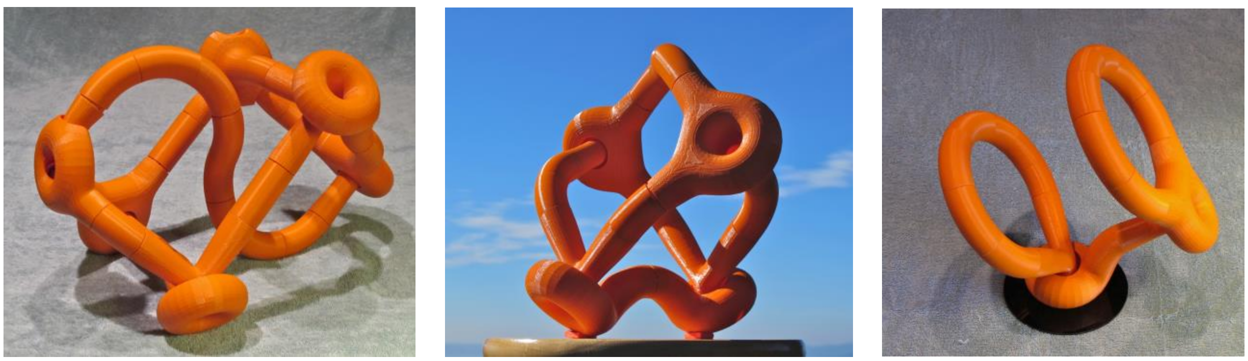
One could also reconstruct the Super-Bottle shown in Figure 5a, by introducing arched connector pieces that bend through 109.47°. However, a more attractive solution reuses some of the smaller curved connectors already introduced into my modular kit. Three connectors bending through 38.96° yield a slightly non-planar connection that forms one branch of a 3-arm hosohedron of genus 2 (Fig.7c). Alternatively, four connectors bending through 30° could achieve the same result.

By making use of the various curved connector pieces, less regular sculptures can also be constructed (Fig.8a,b,c).



(a) (b) (c)

Figure 7: *Sculptures based on cubic graphs: (a) 3-sided prism structure (genus 8), (b) tetrahedral frame (genus 6), (c) 3-sided hosohedron (genus 4).*



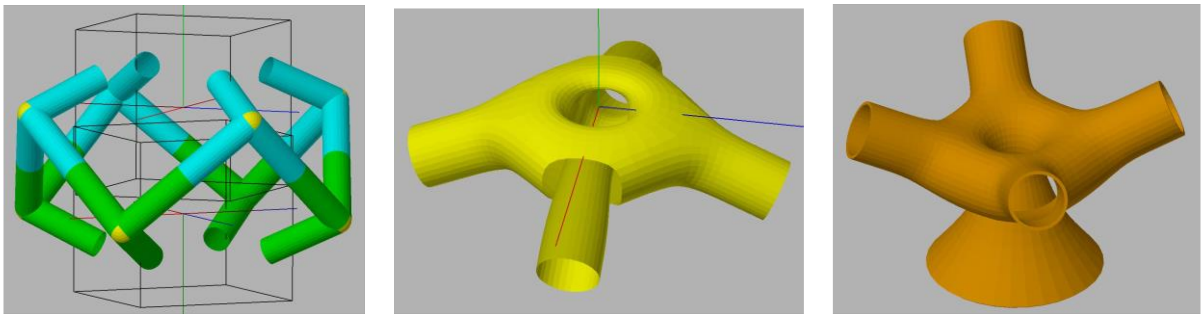
(a) (b) (c)

Figure 8: *Less regular sculptures with different corner combinations: (a) distorted 3-sided prism (genus 8), (b) distorted tetrahedral frame (genus 6), (c) two connected loops (genus 2).*

After having explored various combinations of cube corner modules into Super-Bottles ranging from genus 4 to genus 10, the next plausible goal might be to construct one of genus 12/σ. The most symmetrical polyhedral frame that has the appropriate genus is the pentagonal prism. However, the internal angle of the regular pentagon is 108°, which is larger than the 90° present in the cube-corner component; this would then lead to connecting arms that bend slightly inward. Trying to stay focused on more rounded, convex polyhedral shapes, I chose another approach – introducing a new KBM module that branches into four arms.

6. Introducing a 4-Way-Branch Component

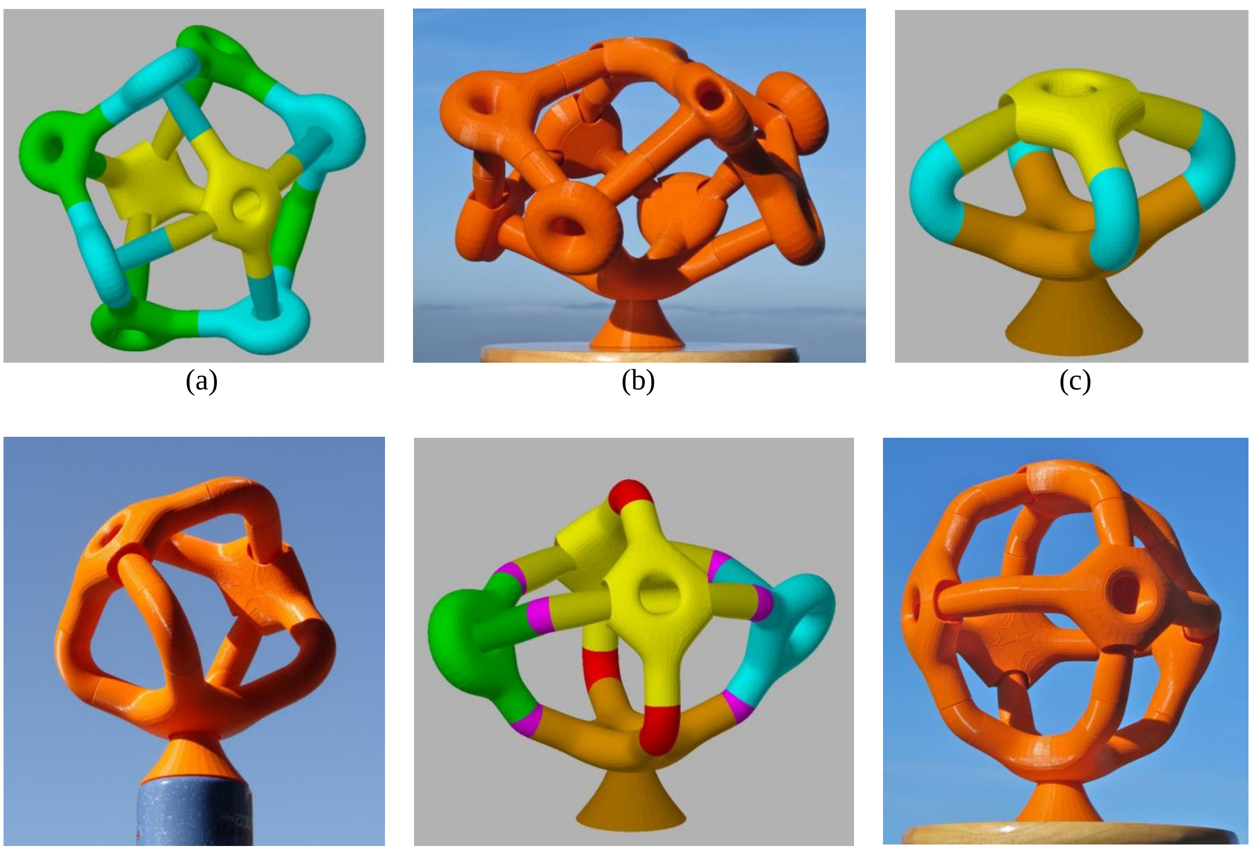
Removing the top and bottom corner modules from the cube frame shown in Figure 5b, leaves a ring of six modules alternatingly tilted by ±54.74° against the 3-fold symmetry axis. By changing this tilt to: arccos(tan 22.5°)=65.53°, one can create a 4-fold symmetrical ring composed of eight cube corners (Fig.9a). The four arm stubs pointing towards the same pole can now be joined by a custom-made 4-way-branch module (Fig.9b). The resulting polyhedral frame has a shape that could be called a "4-sided anti-pyramid." The corresponding sculpture could be balanced on one of the ring edges (Fig.10a). Alternatively, one of the 4-way branching modules can be modified to serve as a stand for the sculpture by fitting a cone shape into the inner hole of the toroidal body (Fig.9c); this module then allows the sculpture to be balanced on this valence-4 corner (Fig.10b).



(a) (b) (c)

Figure 9: *Introducing a 4-branch junction: (a) calculating the branching angle for a D_4-symmetric polyhedron; (b) the resulting 4-arm KBM module; (c) a modified realization that serves as a stand.*

Once the 4-way-branch module became available, I could construct several other attractive polyhedral frames. Two such modules can be connected into a 4-arm hosohedron with curved connectors that bend through 131.06° (Fig.10c). Three 4-way-branch modules can be joined into a doubly-linked ring with connectors that bend through 94.2° (Fig.10d). Alternatively the 3 modules can form a singly-linked ring with connectors that bend through 71.06° and the remaining open arm stubs can be connected to two regular cube corner modules with connectors that bend through 30.24° (Fig.10e). Finally, six of those modules can be connected into an octahedral frame with connectors that bend through 41.06° (Fig.10f).



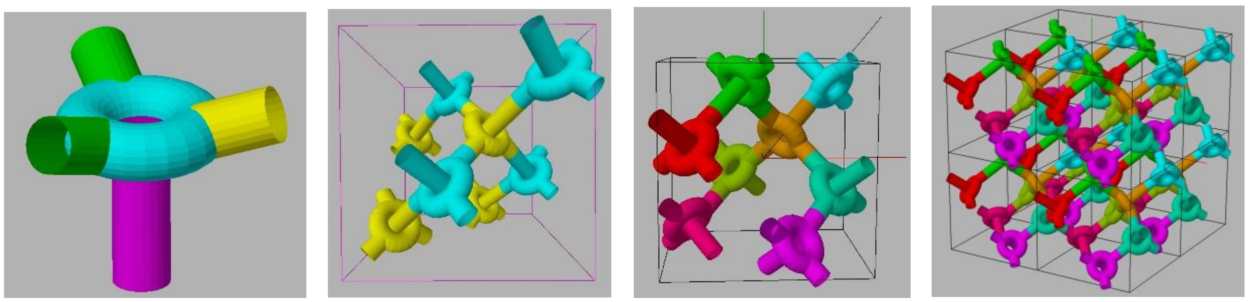
(a) (b) (c)

(d) (e) (f)

Figure 10: *A variety of polyhedral graph structures enabled by the 4-branch junction element: (a) genus 14/σ, (b) genus 14/σ, (c) genus 6/σ, (d) genus 8/σ, (e) genus 10/σ, (f) genus 14/σ.*

7. Klein-Bottle Lattice

A slightly differently shaped 4-way-branch component yields the possibility of making infinitely large lattices. In particular, if the four arms emerge in the four tetrahedral directions (Fig.11a), then we can construct a diamond lattice cell (Fig.11b). Figure 11c shows a more interesting diamond lattice cell in which the eight atomic modules have their "stems" that emerge from the center of the toroidal body point in all eight possible cube corner directions. A 2×2×2 array of such cells is shown in Figure 11d.



(a) (b) (c) (d)

Figure 11: *(a) Tetrahedral 4-way junction; (b) diamond lattice cell comprising eight 4-way junction modules; (c) diamond cell with varied orientations of the KBMs; (d) a 2×2×2 assembly of such cells.*

Figure 12 shows a perspective, close-up view of a larger diamond lattice seen from the crystallographic <1 1 0> direction, revealing the well-known channels in that direction.

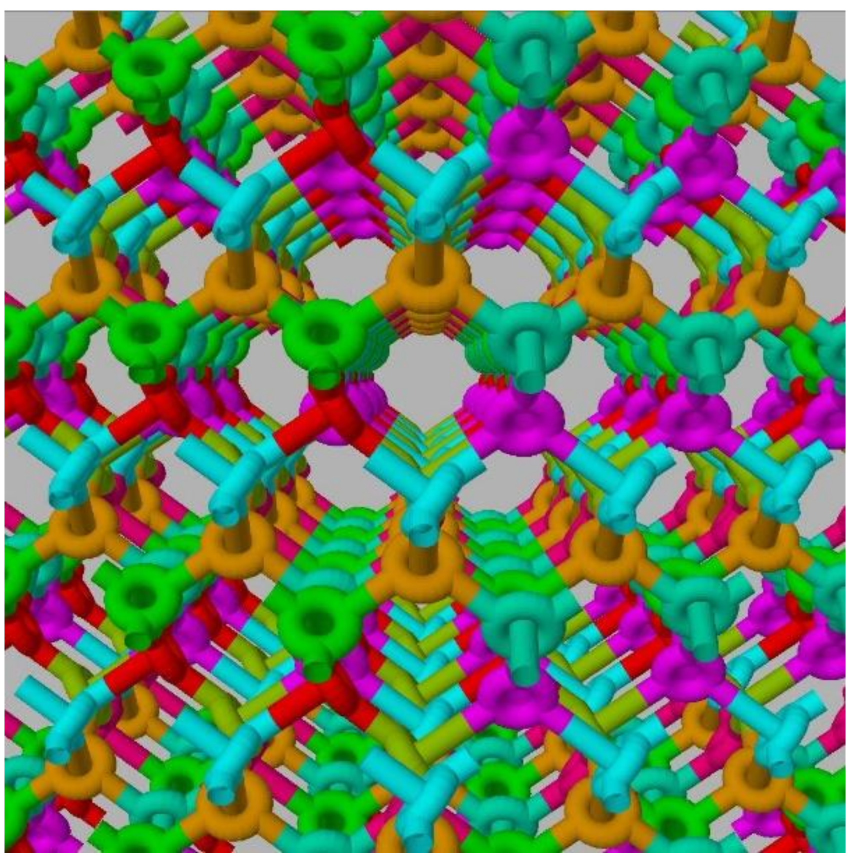


Figure 12: *Klein-Bottle Lattice in perspective view, showing channels in the <1 1 0> lattice direction.*

8. Summary and Conclusions

Stimulated by the LEGO-Knot project [8][9], I designed a set of modular parts that allow me to compose not only various handle-bodies, but also Klein-bottles and single-sided surfaces of higher genus. The key components are 3-way and 4-way tubular junctions combined with a Klein-bottle mouth (KBM) that allows some of the arms to exposes opposite sides of the surface of this module. For most compositions, the resulting surface will be single-sided. In this case, combining t pairs of 3-arm modules with q 4-arm modules permits the construction of a 2-manifold with a genus as high as: $2*(t+q+1)$. This correspond to a connected sum of $(t+q+1)$ Klein bottles – which I call "Super-Bottles." For a few special assemblies of these modules, the resulting surface remains orientable and double-sided ($\sigma=2$); the genus then drops to half the value that it would be for the single-sided Super-Bottle.

The assembly of some of these tubular networks requires the use of curved connector pieces. Unfortunately, most of these connectors must bend through somewhat different angles, and these angles are not related by any simple ratios, so that the required bending angle could be built up in a modular manner from a few generic connector modules. Thus, it would be advantageous to fabricate these connectors out of some flexible material, so that a particular piece could cover a range of angles. An approach using a piece of flexible hose would probably be adequate for purely topological studies – and perhaps even for realizing some "LEGO®-like" building-block set. However, for the construction of aesthetically pleasing sculptural models, a better match in materials and visual appearance between connectors and junction modules is desirable.

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