

Colors and Incomputability

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Abstract

This paper explores some surprising connections between the frequencies of light we observe and foundational questions in the mathematics of real numbers and the theory of computation. We find that these foundational issues imply limitations on what can be seen, separate from any limitations from the laws of physics. Furthermore, it turns out that, given the kinds of observations one can make of light, the most reasonable expectation for the actual frequency of light underlying any set of observations is that that frequency comes from a particularly unusual class of uncomputable numbers called generic real numbers.

Introduction

What would we see if a light wave whose frequency corresponded to an uncomputable number hit our eye?

This question—in a form that will be refined and made more precise—constitutes the springboard for this paper. Given that images are and will continue to be perceived, manipulated, and analyzed by people and machines (analog as well as digital), questions of what can in principle be seen or otherwise observed appear to be both artistic and mathematical questions.

Mathematically, we describe colors with real numbers, and, indeed, our models of reality instruct us to do so. Whether it is the frequency, the amplitude, or phase, our description of the sights we perceive is grounded in the real numbers.

Yet the real numbers are really quite strange. It goes well beyond the familiar uncountability of the real numbers, and into arenas explored in the set theoretical foundations of mathematics, much of which is not part of the general common knowledge of mathematicians. Yet if mathematical logic is the foundation of mathematics, it behooves us, if we are committed to exploring the intersection of mathematics and art, to examine whether this foundation has implications in the realm of understanding color.

Ultimately, this paper will not so much answer the question of what happens if light with an uncomputable frequency is seen, but will rather show how mathematical logic constrains the way one can observe light and what one can conclude from such observations. While it will ultimately turn out that there are a few ways the world can work, we will see that, up to some reasonable assumptions, the frequencies we would most expect to observe correspond to an unusual and rather unfamiliar class of real numbers known as generic reals.

Finally, I remark that this paper is not aiming to study how humans in particular see colors, but rather what the foundations of mathematics and the understanding of real numbers and computability theory tell us about what kinds of observations and perceptions are in principle possible. Aspects of this paper are, to be sure, speculative, but that is the nature of mining a new area. Ultimately, the most important function of this paper, beyond the particular arguments it presents, is to open the exploration of how things like mathematical logic and set theory might play a role in framing what we can be perceived and perhaps other art-related questions as well.

The sort of analysis I present in this paper is of a kind that will likely divide readers into two camps. Some will be intrigued by the demonstration that there are connections that can be drawn between considerations of how colors can be observed and seemingly abstract issues in the foundations of mathematics. Others, however, are likely to view the work here as some sort of philosophical navel-gazing, without material consequence. This paper is not going to eliminate that divide, but I would simply say that having seen how fields as disparate as probability, geometry, group theory, Fourier analysis, and information theory have shown relevance to the understanding of artistic expression, I think it would be a mistake not to spend some time examining ways mathematical logic may have relevance to art. And even if this paper does not convince the doubters, perhaps it will inspire further work that will do so.

Light

Light, as is well known, is made up of photons, the quantum particles that constitute electromagnetic waves [19]. One of the distinctive features of all theories of light—from Maxwell’s classical theory [12] to quantum electrodynamics and the Standard Model [19]—is that photons and electromagnetic waves can be produced with frequencies with any real number values. Despite popular misconceptions, not everything in quantum theories is quantized. Sometimes it is proposed that quantum gravity will discretize space and time, but our one (mathematically) successful quantum theory of gravity, string theory, takes place in an arena of real numbers [1]. Measurements of gamma rays provide no evidence of graininess in the structure of space and time [17]. In short, the world, by every measure we have, involves continuous physical variables parametrized by real numbers, and this includes the possible frequencies of light. Thus, it is worth considering how the properties of real numbers and those of light might be intertwined. (In the concluding section, I will consider briefly the consequences if frequencies and other physical variables are limited to some subset of the reals.)

It is of course well known that color as perceived is not simply a matter of recording the frequency of the photons that hit our retina [16]. But it is the case that the frequencies of the photons hitting our retina provide the raw material out of which our perceptions form. Our focus here is not on the particular workings of the eye and brain, but rather on the possible workings of any kind of visual apparatus, and thus this paper will not make specific reference to the human phenomenon of color.

In addition, we know as well that the specification of colors can be performed in various ways. There are spectral colors, those that correspond to the frequency of an electromagnetic wave. We have frameworks for specifying colors such as RGB and CMYK, designed in the context of human perception, and of course these systems allow for the specification of non-spectral colors. These are all matters outside the focus of this paper, although it is worth noting that in all these systems, the specification of the colors in principle involves specifying one or more real numbers, so considerations akin to the ones of this paper would be expected arise. But for the purposes of this paper, it will be sufficient simply to focus on spectral colors. Furthermore, since frequency appears to be a more fundamental concept than wavelength (a light wave changes its wavelength as it moves from air to water, but its frequency stays the same), I will generally refer to the spectral color (or, more simply at times, the color) of a photon by referencing its frequency.

Finally, why the focus on light rather than sound? It is true that we parametrize sound waves by real numbers. However, here we know that such a parametrization is an approximation that breaks down on the small scale, when the discretization of matter (being built out of atoms and molecules) becomes significant, preventing the formation of sound waves of arbitrarily small wavelength. Light waves—or, more properly, electromagnetic waves—are not limited in this way. Consequently, it is light, not sound, that we will investigate here. And since we will be focusing on spectral colors, as determined by their respective frequencies, we will need to understand something about real numbers. It is to this that we turn next.

Real Numbers

The set of real numbers is strange, far stranger than most acquainted with the basic properties of numbers might realize. Because this subject is not part of the typical Bridges material, I will here summarize the important results and concepts. For the added clarity that arises from some amount of simplicity, I will elide some technical details at points.

The peculiarities of the real numbers (denoted, as usual, by the symbol \mathbb{R}) began to emerge with Cantor's proofs [2] that the number of elements in the set of real numbers—a.k.a. the *cardinality* of \mathbb{R} —is larger than the number of natural numbers, even though both sets are infinitely large. This notion that there are different sizes of infinity is one of the linchpins of the study of real numbers in mathematical logic.

While some of the mathematics we are heading towards is not feasible to derive in this paper, it is worth showing Cantor's second proof [4] that there are infinitely more real numbers than natural numbers, so that the non-mathematicians in the readership can gain some handle on how mathematicians formulate notions of infinities. Suppose that the number of natural numbers and the number of real numbers between 0 and 1 were the same. If that were the case, we could place these two sets in a one-to-one correspondence, leading to a list that would look something like this, provided we write the real numbers between 0 and 1 in decimal form:

$$\begin{array}{lll}
 1 & \longleftrightarrow & .a_1a_2a_3a_4\dots \\
 2 & \longleftrightarrow & .b_1b_2b_3b_4\dots \\
 3 & \longleftrightarrow & .c_1c_2c_3c_4\dots \\
 4 & \longleftrightarrow & .d_1d_2d_3d_4\dots \\
 \dots & \longleftrightarrow & \dots
 \end{array} \tag{1}$$

If this were possible, then the list in the right column would be exhaustive, but, as is easily seen, this is not the case. One need merely to construct a number ρ whose decimal expansion is $\rho = .r_1r_2r_3r_4\dots$ such that $r_1 \neq a_1$, $r_2 \neq b_2$, $r_3 \neq c_3$, and so forth. Since ρ is not equal to any number on the list, it is a real number between 0 and 1 that is not in the ostensible enumeration of those real numbers.¹ Consequently, no exhaustive list of real numbers can be placed in one-to-one correspondence with the natural numbers, and so the size of the set \mathbb{R} must be greater than the size of the set of natural numbers \mathbb{N} . (Note, by the way, that it is not enough to prepend ρ to the list, as then we can simply construct a new number ρ' that is not on this new list.)

We use \aleph_0 (“*aleph nought*”) to denote the size of the set \mathbb{N} , and c to denote the size of the set \mathbb{R} , with the above argument showing that c is strictly greater than \aleph_0 . Something of size \aleph_0 is said to be *countable* (shorthand for “countably infinite”). A natural question that emerged is whether there are other infinities between \aleph_0 and c or not. The conjecture that there were no such other infinities was dubbed the *continuum hypothesis* [3], and, as is discussed below, after about a century, it was resolved.

Before that happened, Church and Turing, using an arithmetization of algorithms akin to the numbering that Gödel used in proving his incompleteness theorems [10], recognized that algorithms could be encoded by natural numbers, indicating that there are countably many algorithms. The significance of this for our purposes is that this means there are only \aleph_0 possible algorithms, and thus only this many real numbers can be fully specified. These real numbers are termed *computable* reals [20][21]. The computable reals include all the real numbers with which you are familiar: the integers, the rational numbers, square roots, π , $\cos(\sqrt{3}e)$, algebraic numbers, and so on—any number for which there exists a finitely specifiable algorithm that allows one to compute that number to arbitrary precision. With only \aleph_0 possible algorithms, there are

¹One must take care to make sure ρ does not wind up being equivalent to one of the other numbers on the list by ending in a non-stop run of 9's.

only countably many (\aleph_0) computable reals. It is then a simple matter of comparing \aleph_0 and c to realize that most real numbers are not computable. This will turn out to matter in what follows.

The resolution of the continuum hypothesis came in the 1960s, in the work of Paul Cohen [8][9]. His conclusion was striking. Much like Euclid's 5th axiom, the continuum hypothesis turned out to be neither true nor false. Rather, there are models of mathematics in which it is true (something actually known to Gödel [11]), but there are also models of mathematics in which it is false, in which there are intervening infinities.

My interest in Cohen's work here, however, is on a notion Cohen introduced in resolving the continuum hypothesis. This is the notion of *generic real numbers*, now often referred to as "Cohen-generic reals," because additional notions of genericity have since been found. Not only are these numbers not computable, but in a certain regard, they are especially unknowable.

A Cohen-generic real is a real number that has no properties that can be specified about it other than statements about finite initial segments of its decimal expansion. (One can equivalently say that the only statements one can make about a Cohen-generic real are statements about its membership in intervals, a fact we will use later, but this decimal version is fully correct and makes it easier for the non-specialist to get a handle on the idea.) The Cohen-generic reals are unlike the real numbers you have presumably encountered, and yet, in some sense, which I will discuss later, despite being so unfamiliar, they are also the most typical real numbers.

Clearly, no computable real number is generic, as we can make statements about the full content of the real number, through the consequences of the algorithm that generates it. But it is more than this. Consider an uncomputable real number that has no odd digits after the billionth decimal place; this number is not generic, even if it is uncomputable, because it has a property that tells you information about something other than some initial digits of its decimal expansion. Indeed, if a number has *any* property that goes beyond a statement about the properties of finite initial segments of its decimal expansion, that number is not generic.

Let me stress that it is not simply that we do not know of any such properties; rather, the number in question *has* no such properties! Despite this, we will see that in certain contexts, they would be the most reasonable frequencies of light to imagine we have detected.

How might generic reals be connected to photons and their frequencies? To this end, we need to think what it is mathematically possible to observe.

Measuring Light

Consider an "eye" or, more generally, some system that detects light, and, for simplicity, we will imagine that it detects a single photon. (As a practical matter, the human retina does react to individual photons, but a single photon is not enough to trigger a visual experience in the human brain.) The question, then, is what happens if the photon detected has an uncomputable frequency. Can we determine its color?

To begin with, we have to realize that we do not know if there is some natural set of units in which to measure frequency, so perhaps what we should really ask is this: suppose two distinct photons are detected, and the ratio of their frequencies is not computable. How does this affect what we can be detected or determined about those photons? For convenient shorthand, however, we will simply refer to uncomputable frequencies, knowing that this more careful formulation lies underneath such phrasing.

So suppose that two distinct photons hit whatever perceptual system we have. The question arises as to whether we are sure to be able to distinguish these photons from each other. Certainly, to discriminate

between any two such photons requires that we have a way to determine if two real numbers are not equal to each other; thus the response for each photon would have to be unique, meaning we would have a way of measuring the frequency associated with each photon. (It is no problem, for our purposes, if there is some non-spectral color that produces the same response; that would, of course, have consequences for vision, and indeed the possibility of this happening in the human visual system is why we can use RGB and CMYK to simulate certain spectral colors. In our case, we are not concerned about what the output would be if, for example, two photons happen to hit the detector simultaneously.)

To understand what can in principle be observed, we turn to the Church-Turing thesis [6][7][22]. The Church-Turing thesis says that the class of functions that can be computed in the kind of mathematical framework that Gödel used in enumerating algorithms are the ones that can be computed by any standard computer, such as a Turing machine. Furthermore, this means that any physical system that can be simulated is no more powerful than a Turing machine or, conversely, if you found a physical system more powerful than a Turing machine, it could not be simulated with a computer.

With this in mind, we turn to the question of vision. Given that eyes and brains or photodetectors and electronic circuits are built of the same materials out of which computers can be built, the general expectation is that any perceptual system would be constrained to perform algorithmically by the same rules that govern computers. In particular, the standard expectation would be that any vision system could be no more powerful in principle than a Turing machine programmed to take some experimental input and calculate the corresponding frequency of the light so detected. But since there are only countably many algorithms a Turing machine can execute, even given infinite time, it would not be possible for any such system to distinguish among the uncountably infinite number of possible frequencies. Indeed, there would be whole collections (with an uncountable number of frequencies each, in any reasonable scenario) of frequencies that, upon hitting the “eye,” would have to give the same result. This means that, irrespective of the physical limitations of how refined a visual sensor can be, mathematics and the Church-Turing thesis alone would require that there be a limitation on how precise any eye or other visual sensor can be. This is a new kind of uncertainty principle, arising not from design flaws, experimental errors, or the peculiarities of quantum mechanics, but from the foundations of mathematics and the nature of any conceivable detection algorithm.²

Of course, it is not necessarily the case that the Church-Turing thesis holds for all physical systems; there are those who believe that the Church-Turing thesis fails, such as Penrose in his notion that human consciousness is not algorithmic and thus not constrained by the Church-Turing thesis [18]. If that is the case, then it would be plausible that we could distinguish among all possible frequencies, but if that were the case, the implications would be much greater than for vision. In particular, if it were possible to measure frequencies to arbitrary precision, we could use this visual system as an engine to perform calculations that would not be possible on a Turing machine or, in practical terms, a standard 21st century computer.

There is no question that I fall into the camp that expects the Church-Turing thesis to be true. But regardless, I find it particularly striking that what a being is in principle able to see—a question essential to the visual arts—should wind up being tied to foundational questions in mathematics and the theory of computation. It turns out, however, that issues of computability and countability are just the start.

Generic Photons

In a previous section, I introduced the notion of Cohen-generic reals. What do these have to do with what we see or observe, whether with our eyes or with machines? It turns out that the connection emerges when we

²This notion can be taken further by considering how the possibility of discriminating colors of light depends on the length of time the eye or other detector can run; this question lies beyond the scope of the present paper, but would invoke a notion of temporal complexity akin to Kolmogorov-Chaitin complexity [5].

take a look at what happens when we observe light. We already know, via the Church-Turing thesis, that we cannot determine the frequency of a photon exactly. But this leaves the question: What can we know about a photon's frequency?

We start with what we can observe about photon. Any observation or measurement of its frequency is inherently limited; all we can hope for from such an observation is a statement that the true value of the frequency lies in some interval of finitely determined numbers.³ In a finite time, of course, we will only obtain finitely many such observations, but even in infinite time, with infinitely many observations, still, the only information we would have would be statements about intervals in which the true value of the photon's frequency sits. Any statement about membership in the observed intervals can be rephrased as a statement about a finite initial segment in the decimal expansion of the frequency. (The equivalence of these two formulations is standard; e.g., see [15].) Indeed, unless it turns out that the range of possible experimental measurements is very different from what experience tells us is possible, it is not only that this is the only information we have; it is the only information we *can* have. For example, it is not reasonable to imagine a measurement that would tell you that some physical quantity represented by a real number had no "7" appearing after the millionth decimal place; this is not what measurements look like.

Now what should we conclude about the actual frequency? Jaynes's maximum entropy principle [13][14] tells us how to handle partial information: in figuring out what we would expect the true result to be in a situation of partial information, we should not imagine any additional constraints beyond what is known. For a simple concrete example, if we were to measure the frequency of some photon to be between 560.83 and 560.84 THz, we of course expect the actual frequency in THz to be a number that begins with 560.83, but Jaynes's principle further tells us that it would be wrong to assert any further properties regarding the ultimate value of the frequency. Now if we think of this in terms of the frequencies of light being detected, this means that we should not expect those photons to have any properties other than the ones that could be identified by observations, i.e., other than statements about intervals in which the true value of the frequency sits.

You will notice that this description is exactly analogous to Cohen's description of generic real numbers. Indeed, while certainly there are both computable reals and also other non-generic real numbers that would satisfy any particular set of observations, these would, following the maximum entropy principle, be peculiar values for the photon frequency to take in this situation. Rather, if we take Jaynes seriously, we should expect the typical result to be some number that not only is consistent with the measurements could be made but also does not have properties that have not and cannot be observed. Since the only kinds of observations we can make about frequencies appear to be precisely the only kinds of statements that one can make about Cohen-generic reals, we would then conclude via the maximum entropy principle that the actual frequency lying behind a set of observations should be a Cohen-generic real.

Thus we have the intriguing situation that the photon frequencies we find should not just correspond to uncomputable real numbers, but should be expected to be Cohen-generic reals. The intersection of the Church-Turing thesis, the maximum entropy principle, and the set theoretic notion of generic real numbers points to the result that, as long as frequencies of photons are parametrized by real numbers, the spectral colors a visual system—including that of a human being—detects should be understood as corresponding to the colors of photons uncomputable Cohen-generic frequencies, even though we cannot actually calculate these values!

To conclude this section, I note that, as alluded to in a previous footnote, if our model of observation is different, we might get different sorts of information about the photon, but that does not change the underlying phenomenon. There are varieties of generic real numbers other than Cohen-generic reals [15].

³This is a simplified version of typical measurements, which provide confidence intervals, for example. The argument can be reworked to accommodate such, but the fundamental connections obtained here would still arise.

As long as the model of measurement is consistent with the Church-Turing thesis, an argument analogous to the one above can be constructed, with the modification that with a different model of observation, one would be led to a different genre of generic reals. The net effect is still that the colors we see should be expected to correspond to unknowable numbers.

Conclusion and Generalizations

What is the upshot of all this? What are the options before us?

On the one hand, might one argue that, given the imprecision inherent in any detection mechanism, that we should not concern ourselves with these questions? Such reasoning strikes me as short-sighted, like arguing that understanding calculus and limits is irrelevant, since we cannot construct infinitesimal objects. The only way to understand if there are perceptual (or other related) consequences arising from foundational issues in mathematics is to explore what consequences there are and to see where that takes us.

So, engaging the questions explored in this paper, what are the conclusions we can reach?

If light can be produced with any real number frequency, but our perceptual devices are constrained by the Church-Turing thesis, it is impossible for our perception to determine the frequency of an arbitrary photon that it detects. In any situation, the perceptual device would only be able to produce countably many outputs (this is even assuming infinite time), leaving the system unable to distinguish among all the possible frequencies. Put another way, given two distinct frequencies impinging on the perceptual device, it is not the case that the “eye” must respond differently; indeed, there must be an uncountable infinity of frequencies that produce the same output. Thus, the Church-Turing thesis implies its own version of an uncertainty principle limiting our perception of colors.

But that is a simple counting argument. What is more intriguing is where the kinds of observations we can make take us. Suppose a photon has been detected, and some information about its frequency has been obtained. What can we infer about the true value of the frequency? In any model of observation that makes contact with what we are actually able to do, Jaynes’s treatment of information and entropy leads us to conclude that we should expect the actual frequency of a photon we observe to correspond to a generic real number—an uncomputable real number of a particularly unknowable kind. Put another way, the colors that we see would most naturally be expected to correspond to such frequencies.

The above picture represents the most conservative possibility. Logically speaking, there are two other consistent, but radical, alternatives. One is that not only can light be produced with any real number frequency, but that also we or devices of our design *can* necessarily distinguish unequal real-valued frequencies. If this is the case, then whatever we are using to detect the frequency of the photon is more powerful than a Turing machine, more powerful than a general purpose computer. It seems unlikely that such an “eye” could exist—if it did, we could not simulate it on a computer!—but it is, in principle, an alternative, as the Church-Turing thesis in the context used here is strictly speaking a conjecture, albeit one many find compelling.

The other is that light cannot be produced with any real number frequency. While we cannot rule out this possibility, there is no evidence that this is the case. We would then have to ask what number system is appropriate for characterizing the frequencies of light (and why). Nonetheless, in this new context, there would still be algorithmic constraints on the observations one could make, resulting in limitations based on Jaynes’s maximum entropy principle on what one could reasonably infer from a set of measurements, and so the chain of argument of the present paper would still apply, even if the particular results looked a bit different. But, most remarkably, this scenario would imply imply that light (and presumably other physical phenomena) should be characterized by parameters chosen from some subspace of the real numbers that is

not complete; such a scenario is sufficiently outside existing formulations of light that we defer consideration of this alternative to a future exploration.

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