Immersion in Mathematics

Judy Holdener
Department of Mathematics and Statistics, Kenyon College
201 North College Road, Gambier, OH 43022, USA
holdenerj@kenyon.edu

Abstract

In this article I describe the meaning of my digital work of mathematical art titled "Immersion."

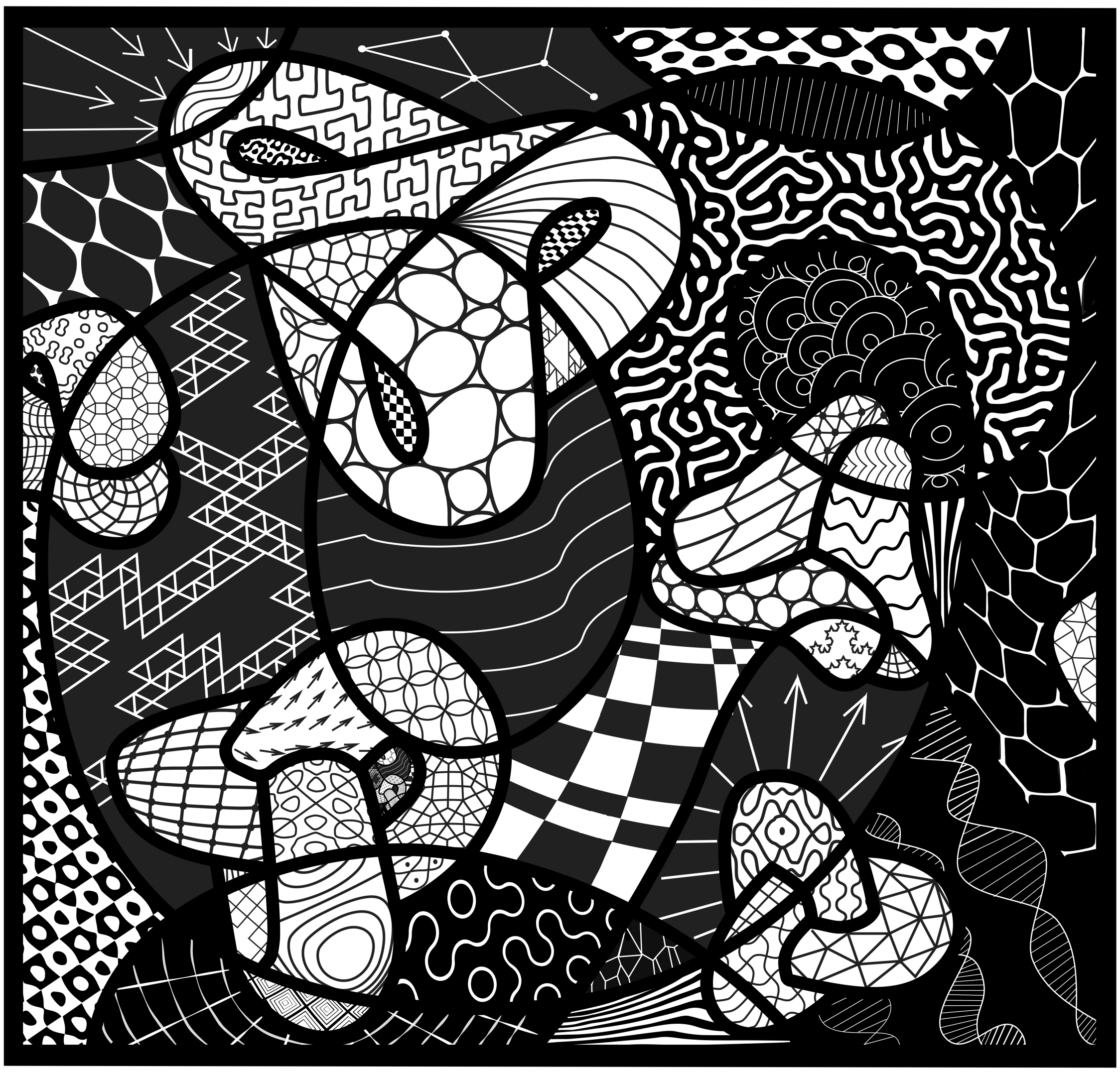

Figure 1 : *"Immersion" by Judy Holdener, 2015* ©

Introduction

Definitions play a fundamental role in the axiomatic structure that characterizes mathematics, and the creation and use of definitions in mathematics differ from those of definitions in our everyday language. In my artwork "Immersion," I examine the notion of "immersion" from both the vernacular and the mathematical points of view. In the first half of the paper I speak about patterns in my artwork that reflect the nature of my own immersion in the world of mathematics – an immersion resulting from my day-to-day work as a mathematician. In the second half of my paper, I tackle the formal mathematical definition of "immersion." As I will explain, it is the formal notion of "immersion" that defines the composition of my work.

As a mathematician, I am immersed daily in a world that is foreign – even scary – to most of the general public. It is my hope to make the nature, content and beauty of mathematics more accessible to a wider audience.

Immersed in the World of Mathematics

As a professor at a liberal arts college, student learning is my number one priority, so many of the patterns in *Immersion* reflect the content of the courses I often teach. I have included patterns involving gradient fields, contours, and normal vectors to reflect my love of multivariable calculus – a course I teach almost every year. I have also included wallpaper patterns and tilings to represent my enjoyment of abstract algebra, an upper-level course containing the language and structures needed for quantifying symmetry and pattern in the mathematical and natural worlds. Readers who attended the Bridges 2014 conference in Seoul might recognize five of the patterns (e.g., the pattern at the lower left corner and another at the upper right) as divergence patterns; they were created by coloring sources in a sinusoidal vector field in white and sinks in black [3]. Finally, several of the patterns in my artwork stem from research I have conducted with Kenyon undergraduates. This work has created patterns and connections between patterns that hold special meaning to me; as such, the work deserves a more detailed coverage. In the discussion to follow, we describe how patterns within "Immersion" reflect a surprising connection residing between two well-known objects in mathematics: the Thue-Morse sequence and the von Koch curve (also known as the "Koch snowflake"). The mathematical details related to this work can be found in two papers I co-authored with former Kenyon students Lee Kennard, Jun Ma, and Matthew Zaremsky [4, 5]. [1]

The von Koch Curve. The Koch Snowflake is a well-known fractal object, defined iteratively starting with an equilateral triangle. At each iteration, we remove the middle third of each edge in the figure and replace it with two line segments that form the sides of an equilateral triangle having base equal to the removed edge. The fourth iteration of one side of the snowflake appears within "Immersion."

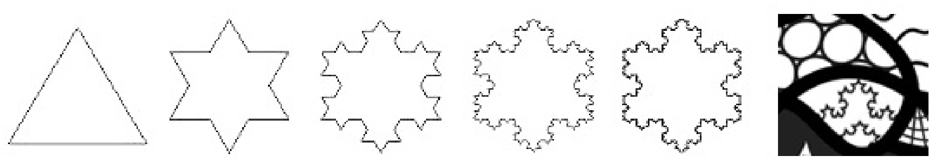

Figure 2 : *A detail of "Immersion" displays the fourth iteration of the von Koch curve*

The Thue-Morse Sequence. I like to think of the Thue-Morse sequence as a sequence constructed by contrarians...or perhaps by a faculty committee. Imagine that a faculty committee wants to create a tile border around their department's coffee room. The first faculty member places a square white tile on the

[1] All three former students now have doctoral degrees (from the University of Pennsylvania, Princeton University, and the University of Virginia, respectively).

wall. The next faculty member comes along and exclaims, “no no no, that is all wrong, the next tile should be black. Seeing a black tile to the right of a white one, a third committee member insists that the next two tiles be black followed by white. A fourth committee member places four tiles (black, white, white, black) in contrast to the first four (white, black, black, white), and the process continues. The result is the one-dimensional tiling depicted in Figure 3.

Figure 3 : *The Thue-Morse one-dimensional tiling created by contrarians.*

Now suppose the same faculty committee wishes to tile the floor of the coffee room. Extending the same contrarian approach to the placement of tiles in two directions results in the tiling given in Figure 4. A transformed version of this same tiling appears within “Immersion.”

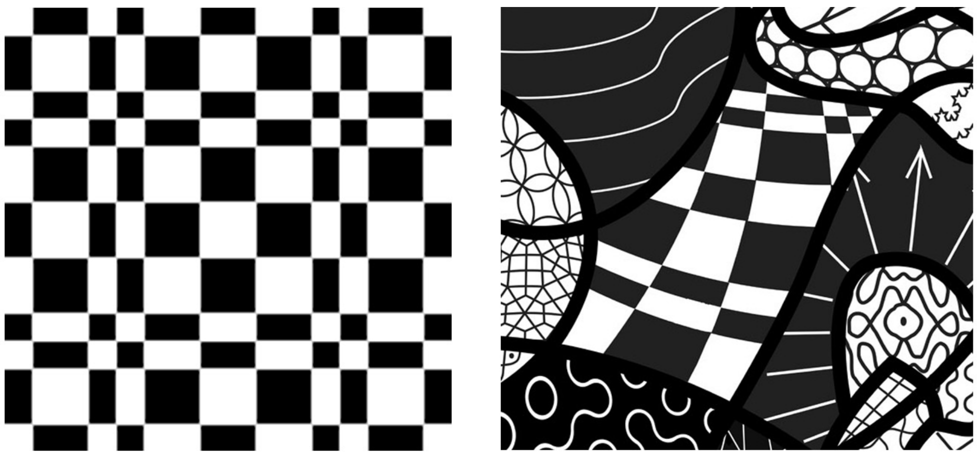

Figure 4 : *A transformed copy of a Thue-Morse tiling appears within “Immersion”.*

More formally, the Thue-Morse sequence is a two symbol sequence typically defined by iterating a substitution map σ. Given the set A^* of nonempty words constructed from the alphabet $A = \{w, b\}$, define the morphism $\sigma : A^* \to A^*$ by setting $\sigma(w) = wb$ and $\sigma(b) = bw$. If $\sigma^0 = w$, we see that σ generates the sequence of words:

$$\{\sigma^n(w)\}_{n\geq 0} = w, wb, wbbw, wbbwbwwb, wbbwbwwbbwwbwbbw, ...$$

which converges to the Thue-Morse sequence:

$$t = \lim_{n\to\infty} \sigma^n(w) = wbbwbwwbbwwbwbbwbwwbwbbwwbbwbwwb...$$

The Thue-Morse sequence was named after the mathematicians Axel Thue and Marston Morse, who discovered the sequence in the same decade (1910-1920), but for different reasons. Like the number π, the Thue-Morse sequence is ubiquitous in mathematics, popping up in unexpected places. It has many interesting properties. First, while the sequence is not periodic, it is *recurrent*, which means there is much repetition within the sequence. In particular, if you pick any string W within the sequence, there exists a length L_W such that any substring of length L_W will contain W somewhere within it. The Thue-Morse sequence is also *cubefree*, meaning there is no substring of the form xxx for any word x constructed with the alphabet $\{w, b\}$ (though there are plenty of squares). Dutch chess master and math teacher Max Euwe stumbled across the sequence in 1929 while trying to construct infinitely long chess games under certain draw rules.

Of course, there is nothing special about the symbols w and b; instead, we may choose to use the alphabet $\Sigma = \{F, L\}$, where F and L represent commands for a turtle in the plane (in the sense of the turtle geometry developed in the early 1980's [1].) In my own attempt to visualize this sequence back in 2000, I let the symbol F represent a forward motion of the turtle in the plane by one unit and L a counterclockwise rotation of the turtle by the fixed angle $\theta = \pi/3$. I examined the turtle programs arising from each iteration of the substitution map σ. In particular, the *Thue-Morse turtle program of degree* n, denoted by TM_n is defined to be $TM_n = \sigma^n(F)$.

$$
\begin{aligned}
&TM_0 = F \\
&TM_1 = FL \\
&TM_2 = FLLF \\
&TM_3 = FLLFLFFL \\
&TM_4 = FLLFLFFLLFFLFLLF \\
&TM_5 = FLLFLFFLLFFLFLLFLFFLFLLFFLLFLFFL
\end{aligned}
$$

The trajectories encoded by these Thue-Morse turtle programs turn out to be surprisingly interesting. Figure 5 shows the results of the Thue-Morse turtle programs of degrees 4 through 7,

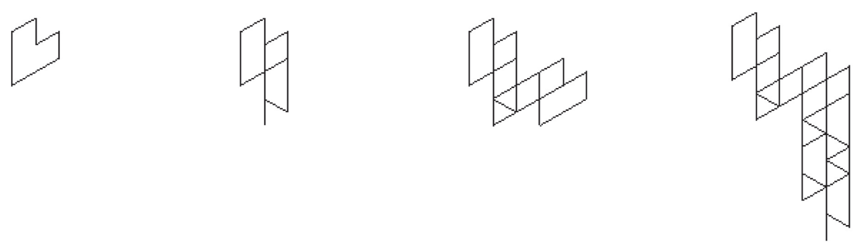

Figure 5 : *Thue-Morse turtle programs of degrees 4 through 7*

and Figure 6 illustrates the turtle trajectory arising from TM_{10}, which generates a second Thue-Morse pattern found in "Immersion."

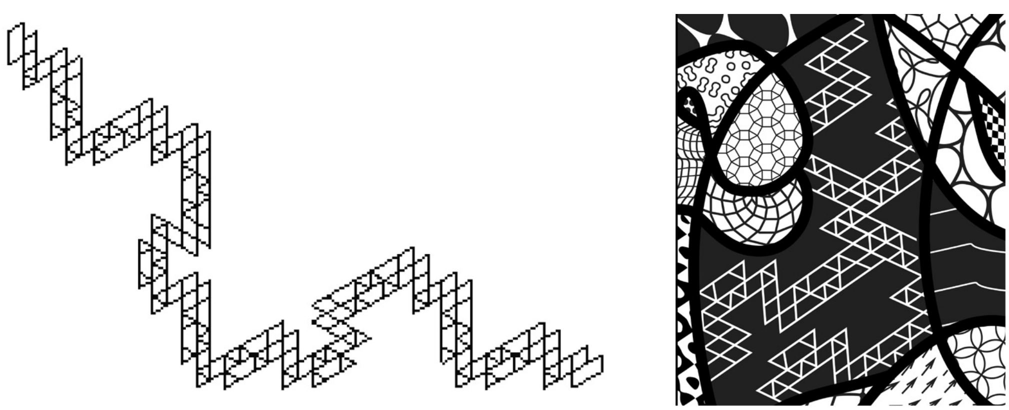

Figure 6 : *The Thue-Morse turtle program of degree 10 generates one of the patterns in my artwork.*

Indeed, the trajectories corresponding to the even terms of the Thue-Morse sequence are starting to resemble the familiar Koch snowflake! Skeptical? Consider TM_{14} in Figure 7.

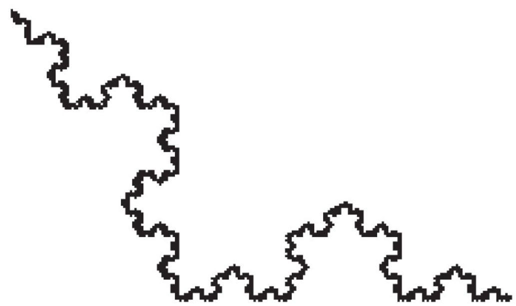

Figure 7 : *The Thue-Morse turtle program of degree 14 resembles one-third of the Koch Snowflake!*

The Mathematical Definition of Immersion

As I have mentioned, the formal mathematical definition of immersion plays a significant role in the composition of my piece. In particular, the white closed surfaces floating in space are different snapshots of a surface known as the "Boy Surface" – named after Werner Boy who first discovered the surface in 1901. Boy discovered the surface when his thesis advisor David Hilbert asked him to prove it was impossible to immerse the real projective plane into three-dimensional Euclidean space. As it turns out, David Hilbert was wrong; Boy's surface illustrates that it is indeed possible to immerse the real projective plane into $\mathbb{R}^3$, and the surface became the focus of Boy's doctoral dissertation [2].

Formally, an *immersion* between two differentiable manifolds M and N is a differentiable function $f : M \to N$ whose derivative is everywhere injective. So an immersion is similar to an embedding, except that an immersion need not be an injective map between the manifolds. Alternatively, an immersion can be defined as a *local embedding*, meaning that for any $x \in M$, there exists a neighborhood $U \subset M$ of x such that $f : U \to N$ is an embedding.

Immersion in the plane. Some of the boundaries of the black and white patterns in my piece are defined by a closed curve in the plane that can be interpreted as an example of an immersion f of the circle S^1 into $\mathbb{R}^2$. (See Figure 8.) Observe that $f : S^1 \to \mathbb{R}^2$ is not an embedding, because the curve self-intersects (twice), meaning f is not injective. However, locally there is a well-defined tangent line at each point on the curve, including the points of self-intersection. Observe that the neighborhood U of x illustrated on the left of Figure 8 embeds into $\mathbb{R}^2$, with image $V = f(U)$ – the neighborhood of $f(x)$ pictured on the right.

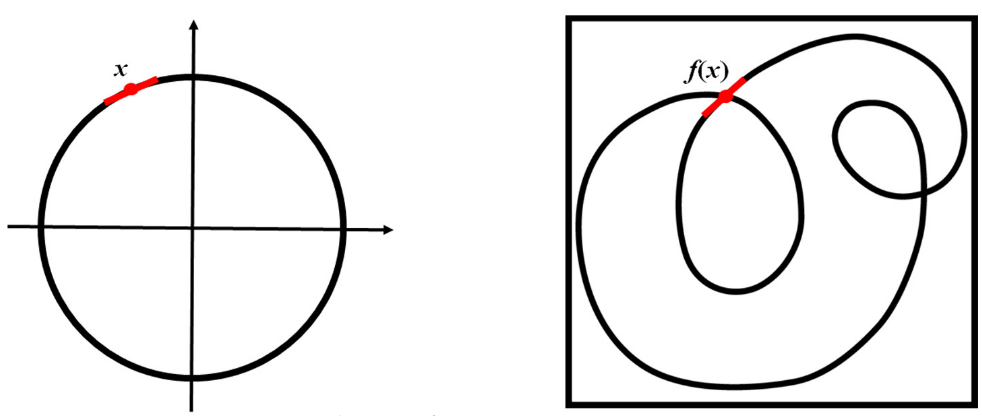

Figure 8 : *A local embedding of S^1 into $\mathbb{R}^2$: the point $x \in U$ maps to the point $f(x)$; there is a single well-defined tangent line at $f(x) \in f(U)$.*

The Real Projective Plane. The real projective plane $\mathbb{R}P^2$ is well-known geometric object to mathematicians, because it is one of the simplest examples of a non-orientable closed surface. (The Klein bottle is another.) The real projective plane also has a significant relevance in art (and, in particular, perspective), because the space can be interpreted as an extension of the ordinary Euclidean plane - obtained by including points at infinity at which parallel lines meet (i.e., the vanishing points).

Classically, the real projective plane is defined to be the space of lines in $\mathbb{R}^3$ passing through the origin. Given the difficulty that comes from visualizing lines in space as points, we typically rely on various surface models to represent the real projective plane. One common model starts with the unit sphere S^2. Since each line through the origin intersects the unit sphere in two diametrically opposite points (a.k.a., the *antipodal points*), we describe $\mathbb{R}P^2$ as the quotient space of the unit sphere obtained by identifying every point $P = (x, y, z)$ with its antipodal point $P = (-x, -y, -z)$. Then every point in the real projective plane is represented twice, so it makes sense to represent the space with the lower hemisphere only (or the upper hemisphere; either one will do), including the points on the equator with antipodal pairs identified. In this way, the hemisphere provides a two-dimensional surface model in which every point determines a unique line in $\mathbb{R}P^2$. Of course, visualizing the identification of the antipodal points in three-dimensional Euclidean space is another challenge (more on that later!).

My own first encounter with the real projective plane was in my graduate-level algebraic topology course at the University of Illinois-Urbana. Starting with the square $[0, 1] \times [0, 1]$ in the plane, I learned to construct the real projective plane by identifying points on the bottom edge with a twist of the top edge, and points on the left edge with a twist of the right edge. To be more precise, point $(t, 0)$ is identified with $(1-t, 1)$ and $(0, t)$ with $(1, 1-t)$ for all $t \in [0, 1]$. A close inspection of the curves in *Immersion* will reveal that my artwork, itself, serves as a model of the projective plane in that the endpoints of the curves heading off of the bottom edge align perfectly with endpoints of the curves exiting a twist of the top edge. Similarly, curves heading off the left edge align with endpoints of curves exiting a twist of the right edge. In this way, my artwork actually depicts two different immersions of the circle into the real projective plane (as opposed to the Euclidean plane, as indicated earlier).

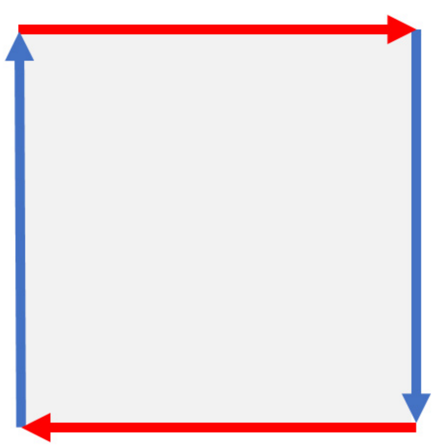

Figure 9 : *Identifying twisted pairs of opposite edges produces the projective plane.*

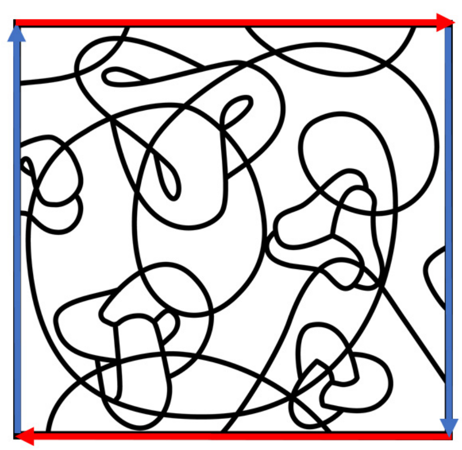

Figure 10 : *Curves from Immersion illustrate a projective plane model.*

Immersion in $\mathbb{R}^3$. It is a well-known fact that the projective plane cannot be embedded in three-dimensional Euclidean space. So if we want to carry out the identification of antipodal points on the equator of the hemisphere modeling $\mathbb{R}P^2$ as discussed above, we must allow the surface to cross over itself. Allowing for self-intersection, there are then multiple well-known models. The Boy Surface featured in my artwork is one of them. (See Figure 11.)

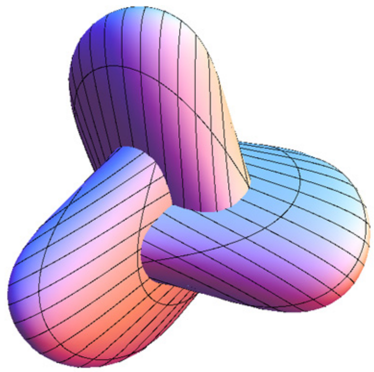

Figure 11 : *The parameterized Boy Surface along side my own rendering.*

Two other well-known realizations of the real projective plane in $\mathbb{R}^3$ are the *Roman surface* and the *cross-capped disk*. These realizations are more degenerate than the Boy Surface in the sense that neither are immersions. As it turns out, the Roman surface contains six "pinch points" at which differentiability fails; see Figure 12. If you imagine that the Roman surface as circumscribed by a tetrahedron, the pinch points are located at the midpoints of each of the six edges of the tetrahedron. The cross-capped disk has two pinch points (at the endpoints of the line of intersection; see Figure 13.)

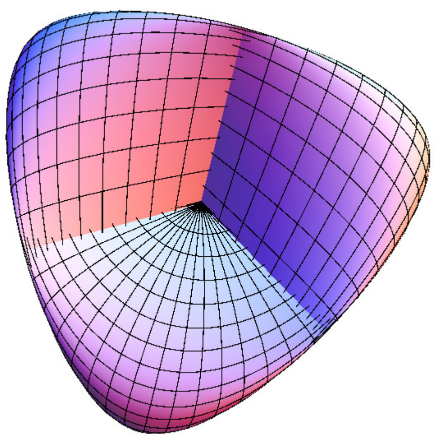

Figure 12 : *The Roman Surface*

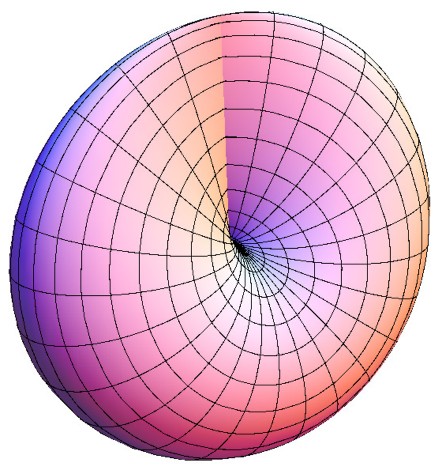

Figure 13 : *The Cross Cap*

Figure 14 below illustrates one way to carry out the identification of the antipodal points on the boundary of a hemisphere model of $\mathbb{R}P^2$ to realize the Boy Surface in $\mathbb{R}^3$. Observe that there are no pinched points or kinks in the final surface. Unlike the Roman Surface and the cross-capped disk, the Boy Surface is differentiable everywhere.

There are five different Boy Surfaces floating in my artwork. The different renditions reflect different viewpoints and different parameterizations that are currently known for the surface. The first known parameterization was provided by French mathematician Bernard Morin in 1978. His graduate student Francois

Apery provided a second parameterization a decade later. Rob Kusner and Robert Bryant (current president of the American Mathematical Society) discovered a third parameterization.

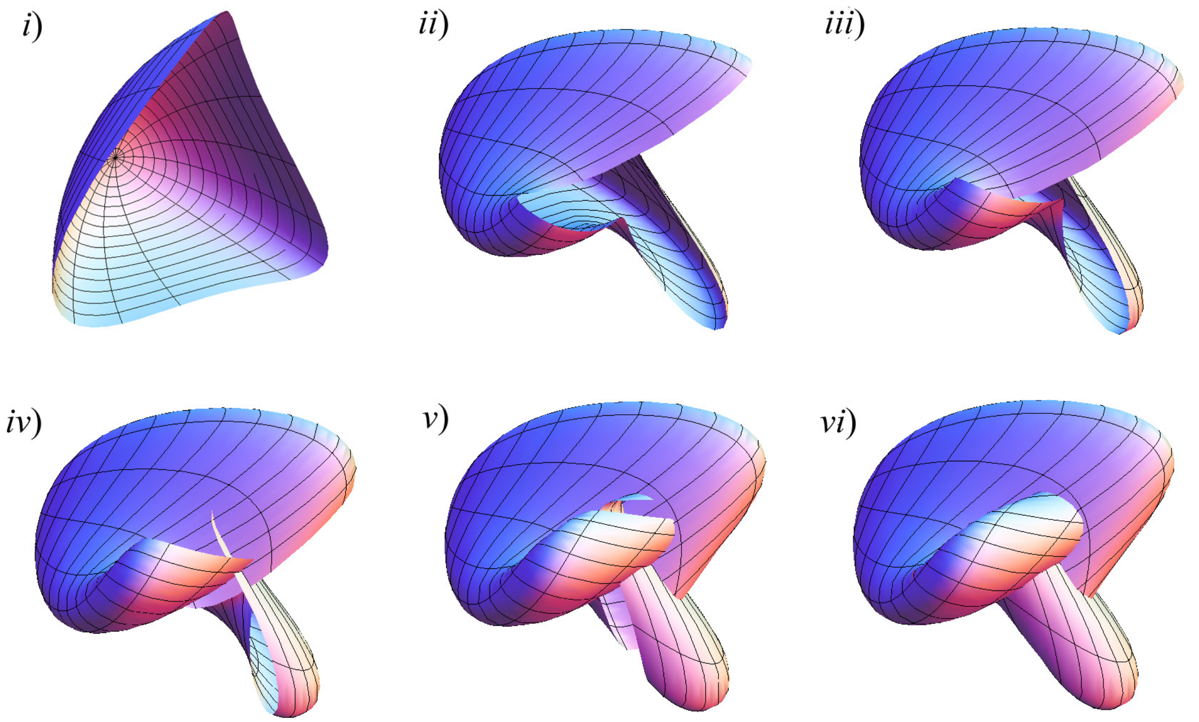

Figure 14 : *The projective plane immersed in 3-space*

Finally, the Boy Surface has served as a source of intrigue for artists other than myself. In 1982 German sculptor Benno Artmann rendered the Boy Surface in his sculpture titled "Ich bin ganz Ohr," and a large sculpture of the surface currently graces the grounds at the Mathematical Research Institute of Oberwolfach in Germany.

References

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