Fractal Gaskets: Reptiles, Hamiltonian Cycles, and Spatial Development

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Abstract

A wide variety of fractal gaskets have been designed from self-replicating tiles. In contrast to the most well-known examples, the Sierpinski carpet and Sierpinski triangle, these gaskets generally have fractal outer boundaries, and the holes in them generally have fractal boundaries. Hamiltonian cycles have been explored that trace out some of these fractal gaskets. Novel solids have been created by spatially developing these gasket fractals over the first several generations. Successive generations are separated in a direction orthogonal to the plane of the gasket, and simple polygons are used to connect the external and internal edges of the gaskets. Since all of the faces in the resulting structures are polygonal, these solids can be described as polyhedra. By varying the spacing between generations, the form of these polyhedra can be varied, creating three-dimensional constructs evocative of architectural forms and geological formations.

Introduction

The Sierpinski triangle and Sierpinski carpet are two well-known classical fractals based, respectively, on an equilateral triangle and a square. Both are created by removing the center tile from a group of tiles that is similar to the starting tile (Figure 1). Such a tile that can be divided into smaller copies of itself is known as a self-replicating tile, or “reptile” [5]. These fractals are formed by iteratively replacing the individual tiles in the reptile with a scaled-down version of the remaining group of tiles. For reptiles such as these, in which the smaller tiles are all the same size, the scaling factor from one generation to the next is the reciprocal of the square root of the number of smaller tiles in the base reptile. E.g., the features of the Sierpinski carpet, which is based on a square divided into nine smaller squares, scale by 1/3 from one generation to the next.

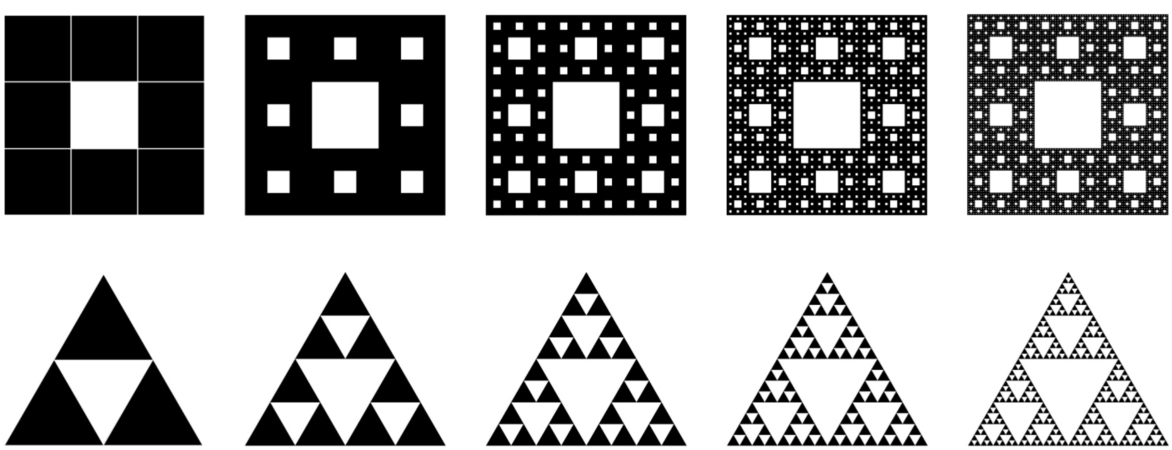

Figure 1: *The Sierpinski carpet and triangle are formed iteratively from reptiles.*

Both these fractals are singly-connected objects, but the Sierpinski triangle necks down to points at each iteration, in contrast to the carpet. Note that both fractals have polygonal boundaries and polygonal holes. The terms fractal gasket and gasket fractal will be used interchangeably here to denote bounded, singly-connected fractals containing holes. Both the boundaries and holes, as will be seen in the next section, can be fractal curves, unlike the fractals in Figure 1.

Other Fractal Gaskets

The Sierpinski triangle is generated by removing the center triangle of four. What if one of the three corner triangles were removed instead? In that case, the resulting group of three triangles (a trapezoid) does not possess rotational symmetry. That means there are three possibilities for orienting each of the three smaller trapezoids, making 27 second-generation structures. If structures that are mirror images are not counted as distinct, there are 15 distinct possibilities, shown in Figure 2.

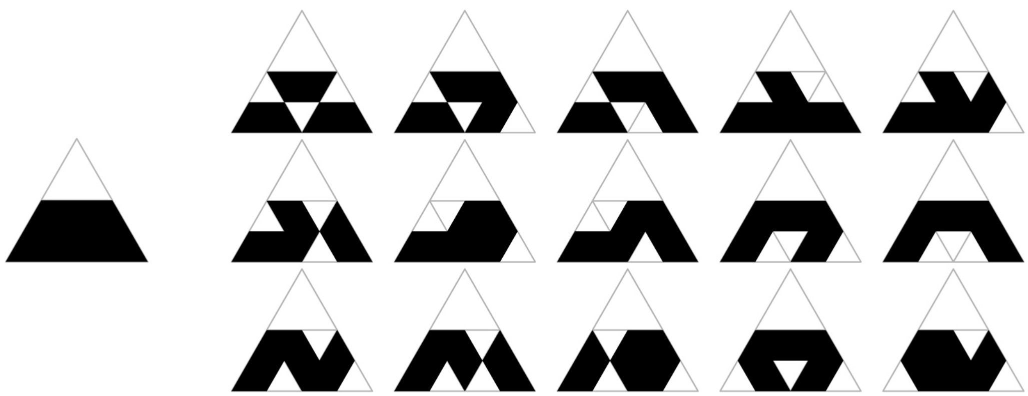

Figure 2: *The trapezoid at left is formed by removing a corner triangle from a group of four. There are fifteen distinct ways of arranging three of these in the next iteration.*

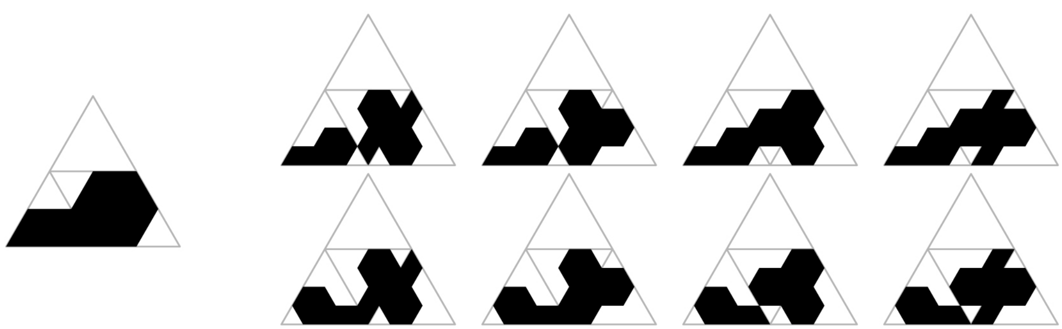

Figure 3: *Each one of the second-generation structures in Figure 2 lacking symmetry can form eight distinct third-generation structures.*

Three of the second-generation structures in Figure 2 have mirror symmetry about a center line, allowing a single choice for subsequent iterations. For the twelve that have no symmetry, however, there are two possible orientations, related by reflection, for each of the three positions, making eight possibilities each. An example is shown in Figure 3. With eight possibilities for each of the twelve second-generation structures lacking symmetry, there are 96 distinct fractals that can be formed. Add to that the three second-generation structures with bilateral symmetry and the Sierpinski triangle, and the four-equilateral-triangle reptile allows 100 distinct fractals based on removing one triangle. All 100 were discovered previously and are shown on a web page by Larry Riddle [1]. Some of them form fractals that are not singly connected or lack holes and therefore don't meet our definition of a gasket fractal. Two examples are shown in Figure 4. A wide variety of fractal forms is possible using the same starting reptile. The fractal of Figure 4b is also seen in Espigulé's work on fractal trees [5].

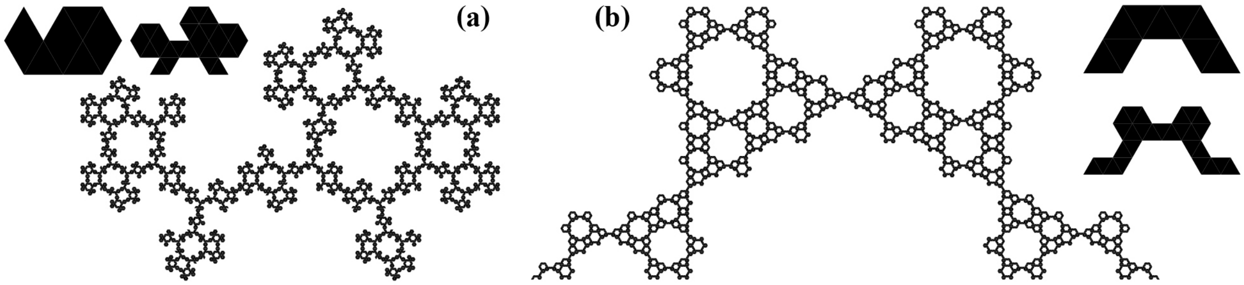

Figure 4: *Two gasket fractals created from an equilateral triangle divided into four smaller triangles. In the first generation a corner triangle is removed. The second, third, and eighth generations are shown here.*

Any square number of squares or equilateral triangles will form a reptile. E.g., while the Sierpinski carpet is based on a nine-square reptile, a four-square reptile can also serve as a basis for fractal gaskets. In this case, there is only one choice for removal of one square. A similar analysis to the above reveals that 232 distinct fractals can be formed using this template, eight of which possess bilateral symmetry. Two examples are shown in Figure 5. Some additional examples can be seen in Reference 2.

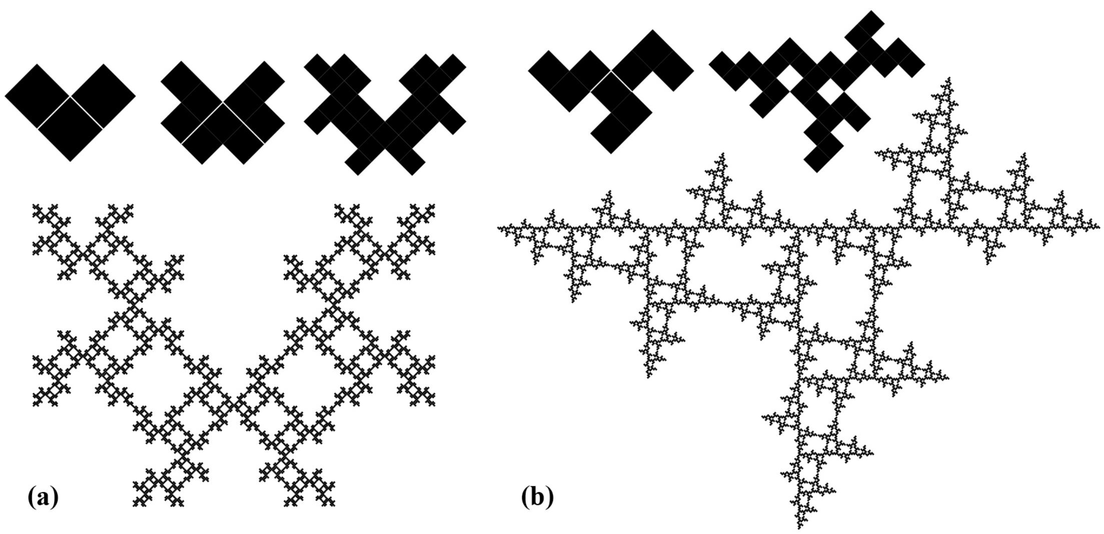

Figure 5: *Two gasket fractals created from a square divided into four smaller squares. In the first generation one square is removed, as shown at upper left, along with the second, third, and eighth generations.*

There are reptiles based on other polygons that can be used to generate attractive gasket fractals, including rectangles, triangles that aren't equilateral, trapezoids, and polyominoes [5]. Examples are shown in Figure 6 for trapezoid and L-tromino reptiles. There are additional reptiles based on trapezoids and trominoes that are divided into 9, 16, etc. smaller tiles [5]. In general, more attractive fractal gaskets result from reptiles with smaller numbers of constituent tiles, as the features scale more gradually between generations. Reptiles can also be employed that are comprised of tiles of more than one size. A kite reptile divided into two kites of one size and three kites of a smaller size was used to generate the fractal gasket of Figure 7. Several dozen examples of additional fractal gaskets based on reptiles can be seen online [3].

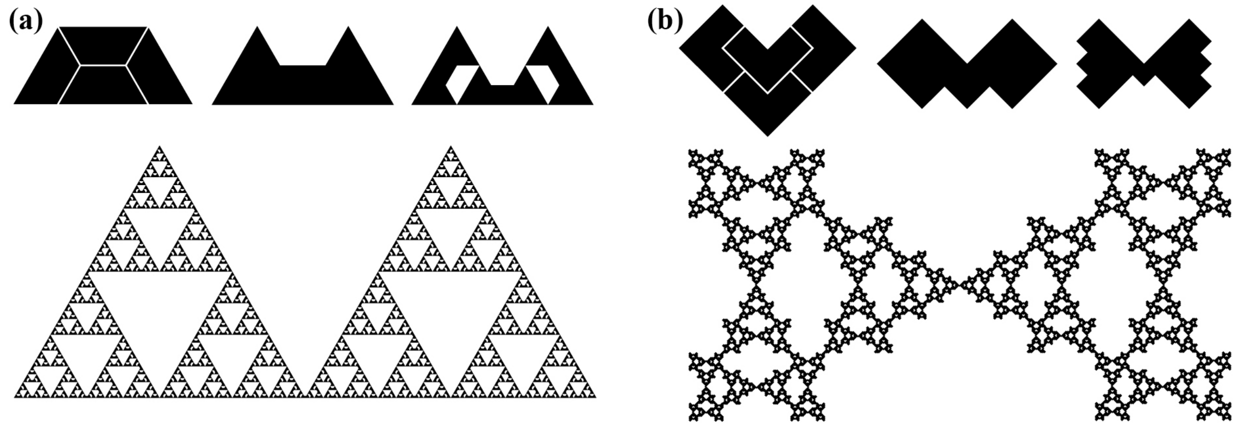

Figure 6: *Trapezoid and L-tromino reptiles and first, second, and eighth generations of fractal gaskets based on them. The left gasket becomes two Sierpinski triangles side by side in the limit.*

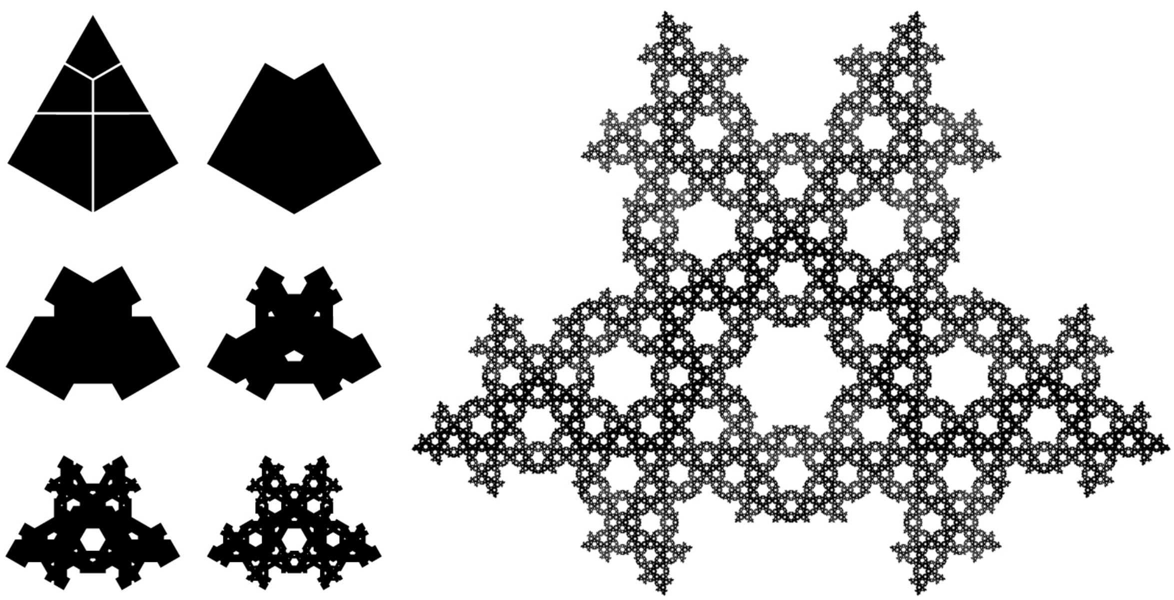

Figure 7: *A kite reptile and the first five iterations of a gasket, along with a larger version of the gasket after twelve iterations.*

Fractal Gaskets and Hamiltonian Cycles

As a mathematical recreation, the tracing out of fractal gaskets using curves has been explored. A specific goal was the use of a single curve to create an object visually similar to common gaskets. In the case of the Sierpinski carpet, points located at the centers of the constituent squares were connected as a Hamiltonian cycle; i.e., with a closed path that visits each point one time. Figure 8 illustrates such a cycle for second- and third-generation Sierpinski carpets. The manner in which nine smaller square structures are connected at the inner corners of the large square hole can be carried out consistently at each iteration, allowing an arbitrarily long path to be created.

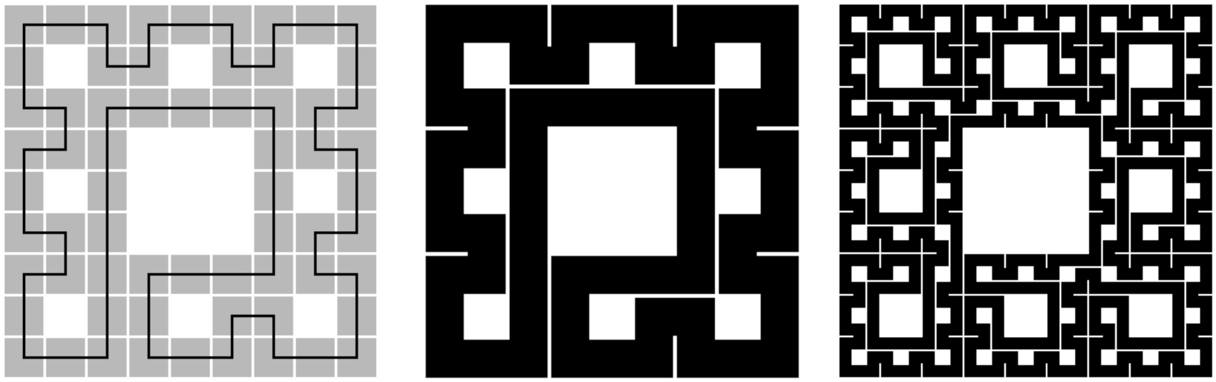

Figure 8: *Hamiltonian cycles for the second and third iterations of a Sierpinski carpet.*

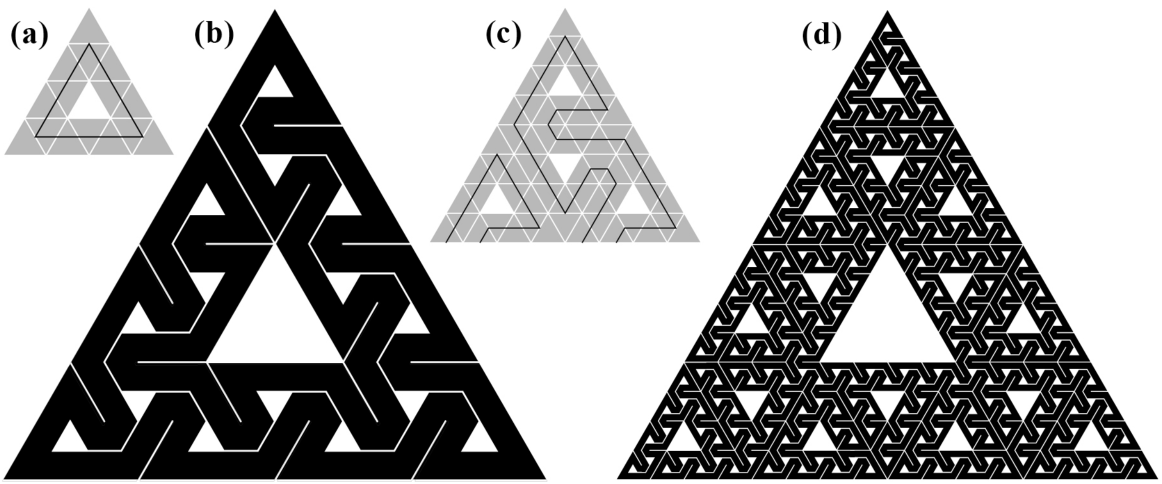

Figure 9: *Hamiltonian cycle for the first (a), second (b), and third (d) iterations of a fractal gasket based on a 16-equilateral-triangle reptile. In c, the upper quarter of the second generation is shown in a manner that better illustrates the construction method.*

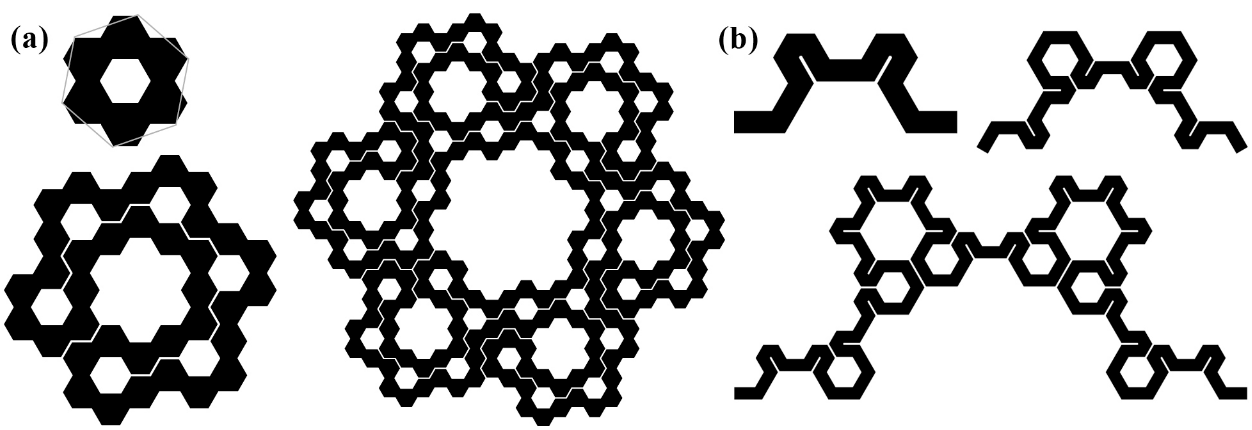

Figure 10: (a) *Hamiltonian cycles for the first three iterations of a fractal gasket based on a regular hexagon. (b) Path for tracing out the fractal of Figure 4b, through three iterations.*

A similar strategy cannot be used with a Sierpinski triangle due to the fact that it necks down to points. It can be used on a triangular analog of the Sierpinski carpet, however, based on a sixteen-triangle reptile. In this case, the points to be connected are located at the midpoints of edges of the constituent triangles, as shown in Figure 9. This cycle can also be iterated as many times as desired.

A hexagonal analog to the Sierpinski carpet can be traced with a Hamiltonian cycle in similar fashion, as shown in Figure 10a. A single regular hexagon is replaced with seven smaller hexagons, though the replacement is not as simple as a reptile. In Figure 10b, the trapezoid-based fractal of Figure 4b is traced out as a path that isn't closed. Two copies of the path can be placed base-to-base to create a close cycle if desired.

Spatially-Developing Fractal Gaskets

Irving and Segerman have created fractal surfaces by spatially separating the iterations of fractal curves and connecting them [4]. If the iterations of a fractal gasket are separated spatially, they can be connected in similar fashion to curves. Unlike a curve, a fractal gasket at a given iteration has finite area. Connecting iterations therefore leads to a solid rather than a surface.

The base of such a solid is the starting reptile. For example, the gasket of Figure 6b begins as a tromino. If the first iteration is displaced in a direction orthogonal to the plane of the tromino, as shown in Figure 11a, simple polygons can be used to connect the two, resulting in a polyhedron (Fig. 11b). This can be iterated by placing three smaller copies of that polyhedron on top of the base polyhedron (Fig. 11c), etc.

In Figure 11, each polyhedral building block is similar. An alternative construction method is to keep the vertical spacing between iterations constant, in which case the shape of the building blocks become increasing stretched vertically. Two examples of this sort are shown in Figure 12, based on the fractal gaskets of Figures 4b and 5a. In the second one, a square base has been added.

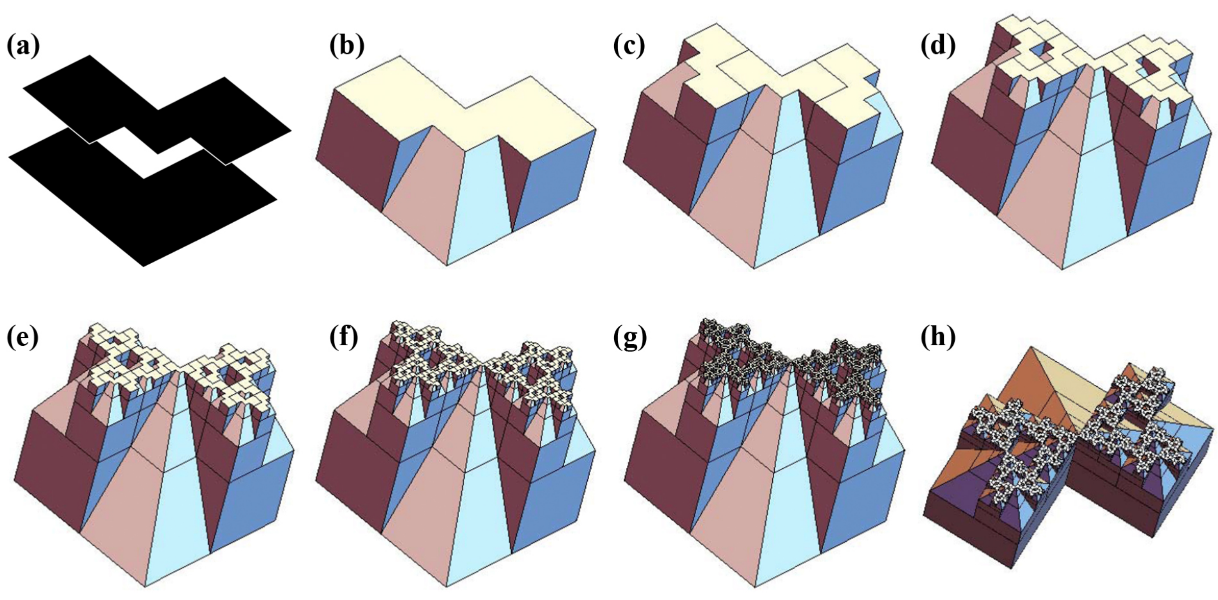

Figure 11: Stages in the construction of a fractal solid based on a gasket. *(a) Starting reptile and first iteration. (b) Polyhedron formed from these. (c)-(g) Next five iterations. (h) Alternate view of sixth iteration.*

An even wider variety of structures can be obtained by varying the height of the polyhedra within a generation. In Figure 13, the middle polyhedron in each group of three is three times the height of the two flanking it. This results in a structure with a tall central peak surrounded by smaller peaks. The symmetry gives this form an architectural look. In contrast, the example of Figure 14 lacks symmetry, resulting in a structure more reminiscent of natural rock formations. Finally, Figure 15 shows two 3D prints based on structures of this sort.

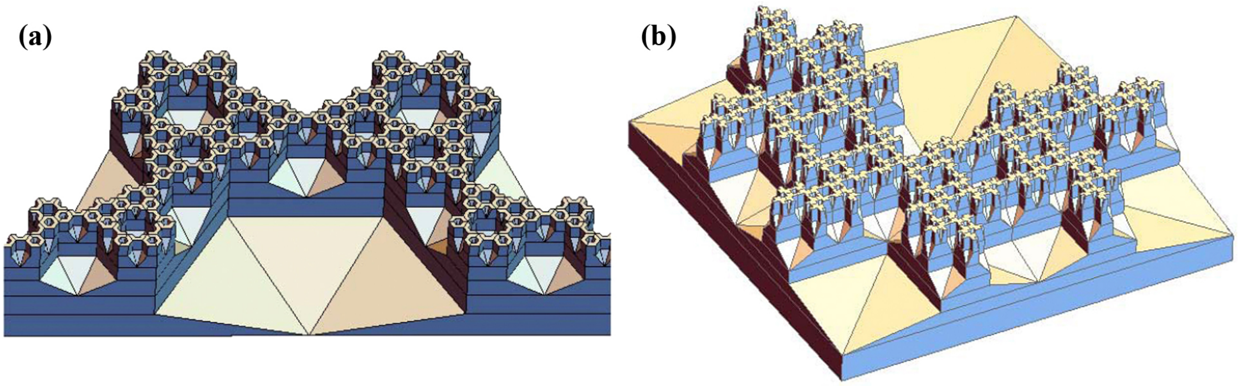

Figure 12: Two spatially-developed fractal gaskets with equal spacing between iterations, carried through six generations.

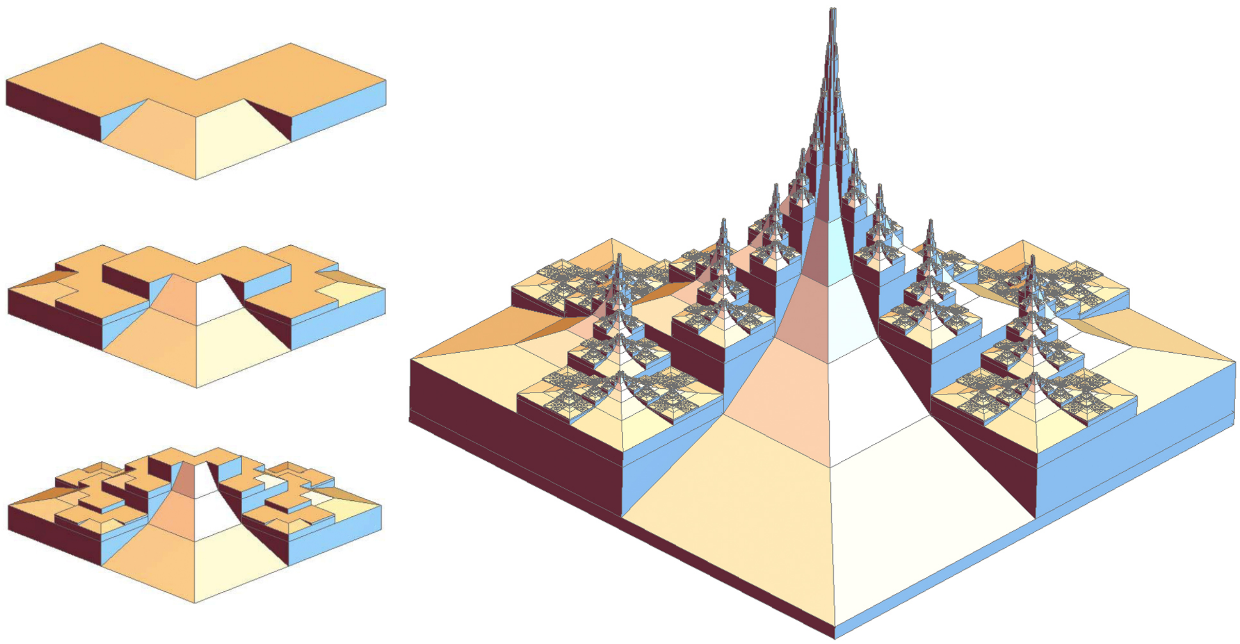

Figure 13: A spatially-developed fractal gasket in which the spacing is greater in the center than the edge polyhedra in each group of three. The first, second, third, and eighth generations are shown.

Conclusions

There is an enormous number and variety of fractal gaskets, and a wide variety of ways they can be employed in two and three-dimensional forms. As a result, there are many possibilities for employing fractal gaskets in mathematical art that is geometric in character or evocative of the natural world. The sort of novel polyhedral forms of Figures 11-15 could possibly inspire architectural designs.

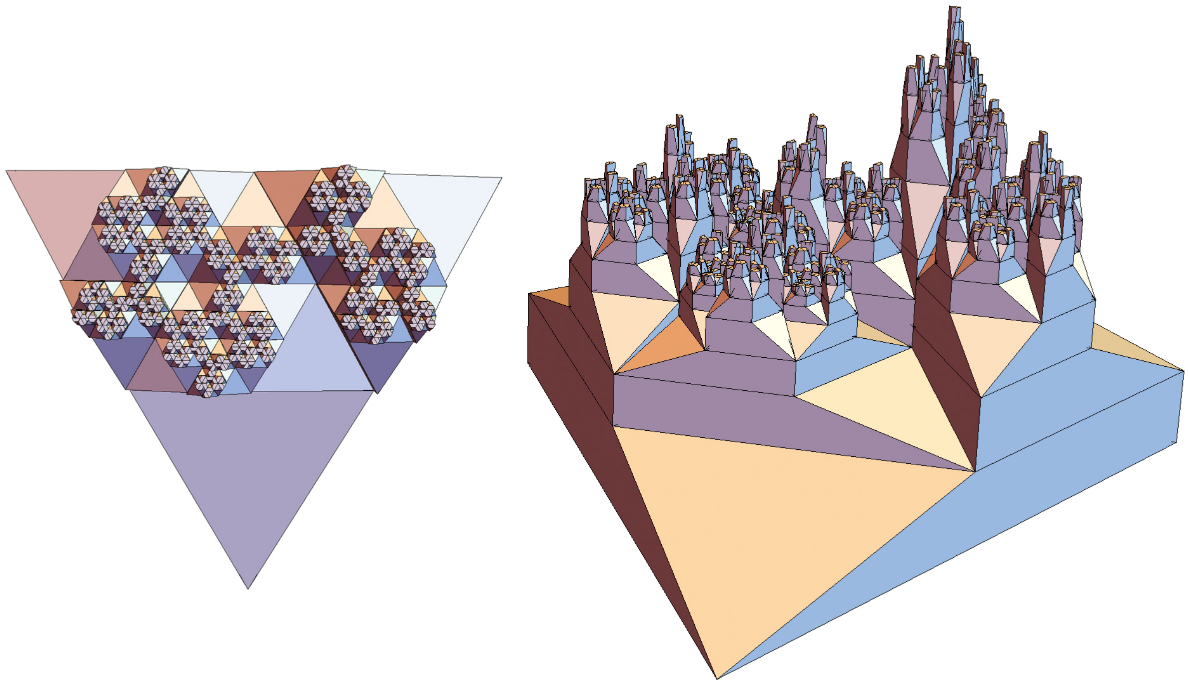

Figure 14: *A spatially-developed fractal gasket carried through seven generations, shown from two viewpoints.*

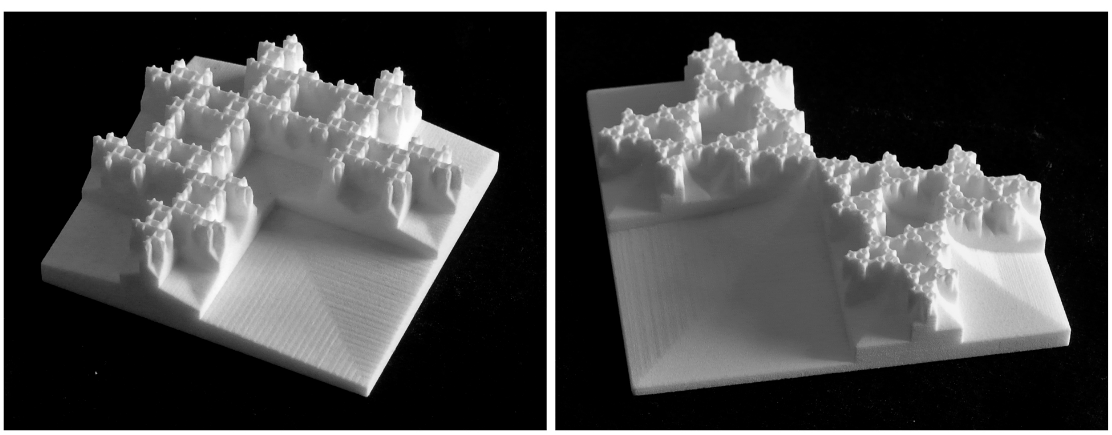

Figure 15: *Photographs of two 3D prints based on the structures of Figures 11f and 12b.*

References

[1] L. Riddle's website; http://ecademy.agnesscott.edu/~lriddle/ifs/siertri/gallery.htm.

[2] L. Riddle's website; http://ecademy.agnesscott.edu/~lriddle/ifs/siertri/boxVariation.htm.

[3] R. Fathauer's website; http://mathartfun.com/fractaldiversions/GasketHome.html.

[4] G. Irving and H. Segerman, *Developing Fractal Curves,* J. Mathematics and the Arts, Vol. 7, pp. 103-121 (2013).

[5] Wikipedia; https://en.wikipedia.org/wiki/Rep-tile.

[6] B. Espigulé Pons, *Unfolding Symmetric Fractal Trees*, Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture (2013).