Some Girihs and Puzzles from the *Interlocks of Similar or Complementary Figures* Treatise

Reza Sarhangi
Department of Mathematics
Towson University
Towson, MD 21252
E-mail: rsarhangi@towson.edu

Abstract

This paper is the second one to appear in the Bridges Proceedings that addresses some problems recorded in the *Interlocks of Similar or Complementary Figures* treatise. Most problems in the treatise are sketchy and some of them are incomprehensible. Nevertheless, this is the only document remaining from the medieval Persian that demonstrates how a girih can be constructed using compass and straightedge. Moreover, the treatise includes some puzzles in the transformation of a polygon into another one using mathematical formulas or dissection methods. It is believed that the document was written sometime between the 13th and 15th centuries by an anonymous mathematician/craftsman. The main intent of the present paper is to analyze a group of problems in this treatise to respond to questions such as what was in the mind of the treatise's author, how the diagrams were constructed, is the conclusion offered by the author mathematically provable or is incorrect. All images, except for photographs, have been created by author.

1. Introduction

There are a few documents such as treatises and scrolls in Persian mosaic design, that have survived for centuries. The *Interlocks of Similar or Complementary Figures* treatise [1], is one that is the source for this article. In the *Interlocks* document, one may find many interesting girihs and also some puzzles that are solved using mathematical formulas or dissection methods. *Dissection*, in the present literature, refers to cutting a geometric, two-dimensional shape, into pieces that can be rearranged to compose a different shape. *Girih* is the fundamental region for tiling of ornamental designs that will be addressed in section 3.

In this paper a set of problems from this treatise are analyzed to trace its' author's mind to answer questions such as how the author constructed the diagrams, whether his solutions are mathematically provable, and whether the conclusions are correct. Moreover, there are diagrams in the document without any explanation for which the present work will offer some mathematical justifications.

2. Interlocks Treatise's Puzzles and Mathematics Problems

This section is devoted to some puzzles in the treatise that present ways that are utilized to transform a geometric shape into another.

2.1. Hexagram (or 6/2 star) and triangle transformations. A puzzle that is illustrated in the treatise is a connected 6/2 star polygon, hexagram, to an equilateral triangle as is presented in the left image of Figure 1. The two images on the right show the details of the dissection. Cutting along the dashed lines in one of the two polygons of the hexagram and the triangle and rearranging pieces, one can construct the other one.

The following is a solution offered by the author of this paper that proves the validity of this transformation:

Solution: Consider the consecutive vertices of *A, B, C, D,* and *E* in the hexagram in the middle image of Figure 1. Each angle of $\angle DCH$ and $\angle HCB$ is 30°. But since angles $\angle ECD$ and $\angle CED$ are each 30° then the two right triangles ΔCHD and ΔCHB can cover the triangular space between *C, E,* and *D* completely without gaps or overlaps. So transforming the hexagram into an equilateral triangle is a straightforward process.

To perform the reverse transformation consider the given equilateral triangle ΔRST and construct the median RN. Construct the perpendicular bisector of SN to intersect the angle bisector of $\angle RST$ at P. Perform the same process on segment NT. Dissect the triangle according to the constructed segments and rearrange the pieces to create the hexagram. The proof is trivial and stems from the construction steps.

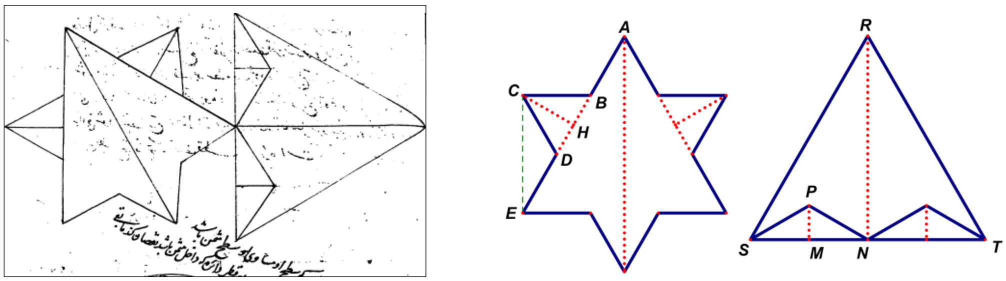

Figure 1: *Transforming a 6/2 star polygon to an equilateral triangle and vise versa*

2.2. Finding the side of a square with the same area as a given octagon. The regular octagon *ABCDEFGH* is given. Find the side of a square that has the same area as this octagon.

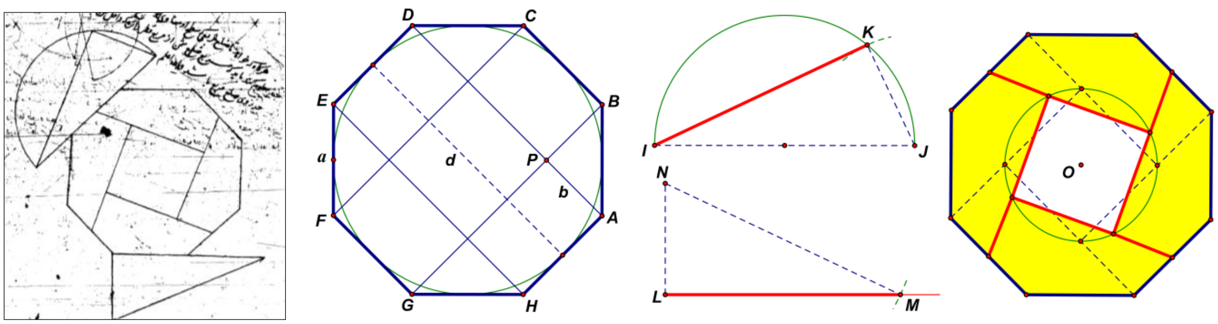

Figure 2: *Finding the side of a square that has the same area as the given octagon*

Solution suggested by the treatise: Find d^2, the square of the diameter of the incircle, which is inscribed inside of the octagon. Subtract a^2, the square of a side of this octagon from it. The reminder, $d^2 - a^2$, is the square of a side of the desired square.

Sketch of proof: The above solution appears in the treatise next to an image of the octagon with two attached triangles (the left image in Figure 2). Moreover, there are cutting lines in this image that divide the octagon. There is no proof for the problem and no explanation for the triangles and cutting lines inside of the octagon in this document. So after proving the validity of the solution, a way of understanding these extra shapes is presented that, perhaps, was in the mind of the author of the treatise.

In the second image from the left in Figure 2, the given octagon is divided into nine pieces. If the side of the octagon is a units, then there is a square with sides a units in the middle, four rectangles with sides a and b units, and four isosceles right triangles with the hypotenuse a and sides b units. From the right triangle ΔABP with angles 45° one obtains $b = \frac{\sqrt{2}}{2}\, a$. If the diameter of the incircle is d units then $d = a + 2b = (1 + \sqrt{2}\,)a$ yields to $d^2 = (3 + 2\sqrt{2}\,)a^2$. Hence one concludes that $d^2 - a^2 = (2 + 2\sqrt{2}\,)a^2$. On the other hand from the division of the octagon into pieces one obtains:

$$\textit{The area of the octagon} = a^2 + 4(a.b) + 4\left(\tfrac{1}{2}b^2\right) = a^2 + 2\sqrt{2}a^2 + a^2 = (2 + 2\sqrt{2}\,)a^2.$$

Therefore, the area of the octagon is $d^2 - a^2$ and the presented solution is valid. □

A note to justify the existence of the two triangles in the treatise's diagram: The two triangles ΔIJK and ΔLMN demonstrate how a side of the aforementioned square can be obtained geometrically from the given octagon: To construct ΔIJK where segment *IJ* is *d* units, one makes an arc with center *J* and radius *a* units to cut the semicircle with diameter *IJ* at *K*, then *IK* is the side of the desired square. To construct ΔLMN let *LN be a* units and the angle with vertex *L* is a right angle. Make an arc with center *N* and radius *d* units to find point *M*. Segment *LM* is the solution.

Please note that there is an infinite number of ways for the division of the octagon similar to what is offered by the treatise as is shown in the left image in Figure 2. The right image is the unique solution for the case that the division lines pass through the midpoints of the sides and the square in the middle has side *a* units.

2.3. Dividing a triangle into four triangles with congruent areas in a way that three of them share vertices with the original triangle. Triangle ΔABC is given. Divide this triangle into four triangles, ΔABE, ΔBCF, ΔCAD, and ΔDEF, in a way that they have congruent areas.

Incorrect solution suggested by the treatise: Suppose *L*, *M*, and *N* are the midpoints of the sides of ΔABC. Construct ΔLMN. Find its incircle. From each vertex of ΔABC make a tangent to this circle to create four triangles ΔABE, ΔBCF, ΔCAD, and ΔEFD. The obtained triangles have congruent areas.

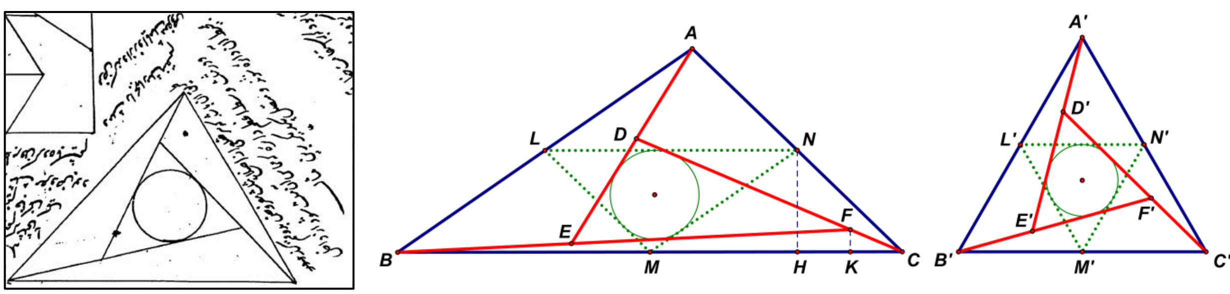

Figure 3: *Dividing a triangle into four triangles with congruent areas*

The solution is invalid: The four triangles ΔALN, ΔLBM, ΔMNC, and ΔLMN are congruent and similar to ΔABC. According to the solution presented by the treatise the areas of the two triangles ΔBCF and ΔMCN should be the same since each of the areas is supposed to be one quarter of the area of ΔABC. The area of ΔBCF is ½ *BC.FK= MC.KF*. The area of ΔMCN is ½ *MC.NH*. Consequently, one should obtain ½ *NH* = *FK*. But in Figure 3 one observes that ½ *NH* > *FK*. Therefore, in general this solution is incorrect.

A valid case: Consider the special case of the equilateral triangle $\Delta A'B'C'$, which is demonstrated in the right image in Figure 3. The two triangles $\Delta L'M'N'$ and $\Delta E'F'D'$ are congruent and therefore, the area of $\Delta E'F'D'$ is ¼ of $\Delta A'B'C'$. Since the three triangles $\Delta A'B'E'$, $\Delta B'C'F'$, and $\Delta C'A'D'$ are congruent and the area of $\Delta E'F'D'$ is ¼ of $\Delta A'B'C'$, the area of each triangle $\Delta A'B'E'$, $\Delta B'C'F'$, and $\Delta C'A'D'$ is ¼ of $\Delta A'B'C'$ as well. So the problem is valid for the special case of the equilateral triangle.

2.4. Octagon and square transformations using dissection method. The octagon *ABCDEFGH* is given. Divide it into four quadrilaterals, four isosceles right triangles, and a square in order to be able to rearrange the pieces to construct a square.

Solution offered by this paper: There is no instruction about this transformation in the treatise other than presenting the left image in Figure 4. So the following argument is based on this image only. Consider an arbitrary point *T* on *GF* and construct *AT*. Rotate *AT* about the center of the octagon to construct the square *KJIL*. Construct square *NMPO* from the midpoints of the sides of this square. Then the octagon can be divided into *AKCB* and three congruent polygons to *AKCB*, right isosceles triangle Δ*NOL* and three congruent triangles to Δ*NOL*, and the square *MPON*. Rearrange the four quadrilaterals to create a square with sides congruent to *AK*+*CK*. Then it is necessary that the other five pieces in the octagon cover the remaining blank space inside of this square without gaps or overlaps.

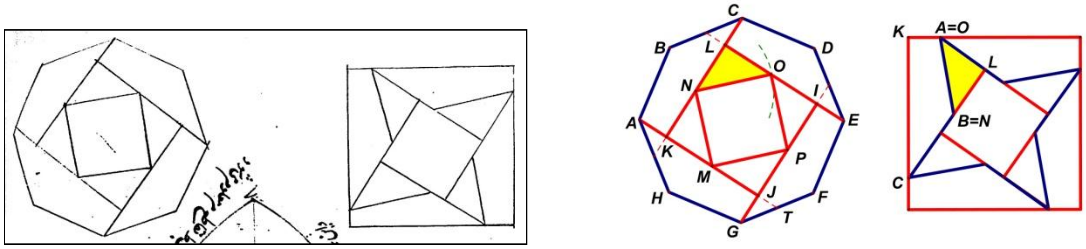

Figure 4: *Transforming an octagon into a square and vice versa*

Obviously since for any point *T* the size of the sides of the large square in the right will change (and as a result the area of this square will change) the problem should have a unique solution in a way that the other pieces can cover the blank space inside of the square completely without gaps or overlaps. If *T* is the unique solution then *AB* and *NO* should be congruent as is presented in the large square on the right image. One guess could be that the midpoint of *GF* is the solution. It seems that if we make an arc with center *N* and radius *AB*, the circle passes through a point very close to *O* but not exactly at *O*. Using this guess one may transform the octagon into an approximate square. Nevertheless, to either accept or reject this solution one needs a mathematical proof. Reversing this transformation is possible if one can find *C* on the right image. The reader may consider this problem as an open question.

3. Interlocks Treatise's Girihs

In this section a number of interesting girihs, which are recorded in the treatise, are presented and analyzed.

Girih (or *aghd*) is a word in Farsi that is used in traditional Persian architecture to refer to the fundamental region of a mosaic design. This word has many other meanings in Farsi, including knot, complexity and connections, and also a unit of measurement. In general, a girih can be extended in all directions to cover a surface using translation, vertical and horizontal reflections, rotations, or a combination. In some cases a girih is self-contained and is used as is constructed geometrically without any extensions.

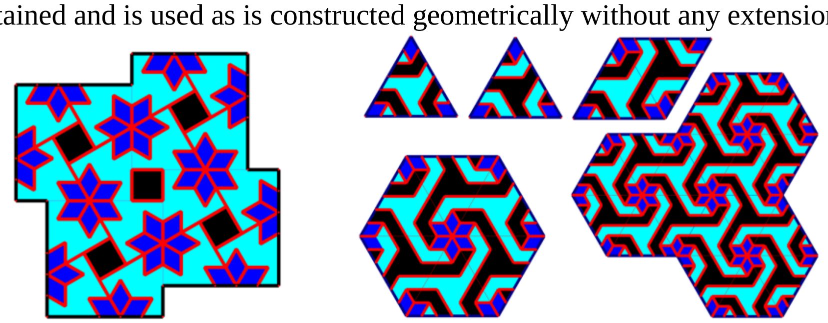

Figure 5: (L) *Extension of a girih by translation, and* (R) *by rotation*

For example, the left image in Figure 10 demonstrates a girih in the *Interlocks* treatise. The right image in Figure 10 shows a tiling that is created based on the extension of this girih, using reflections. However, for the same girih, one can compose a tiling by the extension of the girih, using a translation, as is presented in the left image of Figure 5. The top three images in the middle of Figure 5 exhibit a triangular girih, the same girih but in opposite colors except for the corners, and a colored rhombus girih made from the two triangles. This rhombus cannot be constructed using two copies of the same colored triangular girih. The bottom image in the middle is a hexagon created from a 3-fold rotational symmetry about one of the vertices of the obtuse angle of the rhombus. The image in the right is the final step that is created by the 3-fold rotational symmetry of this hexagon about one of its vertices. This girih without color can be found in the Maher-al-Naghsh's book, *Design and Execution in Persian Ceramics* [2].

3.1. An Interesting heptagram based tiling: The left image in Figure 6 is from the treatise. It presents a girih that includes one half of a regular heptagram on each side of the square. The two images in the middle of Figure 6 show how a heptagram can be constructed from a 7/2 star polygon. The right image is the tiling that is formed using this girih but with some dropped segments. An observer may notice that the tiling includes heptagrams (7/2 star), concave octagons, four of which create a four-fold rotational symmetric shape, and a polygon similar to a maple leaf.

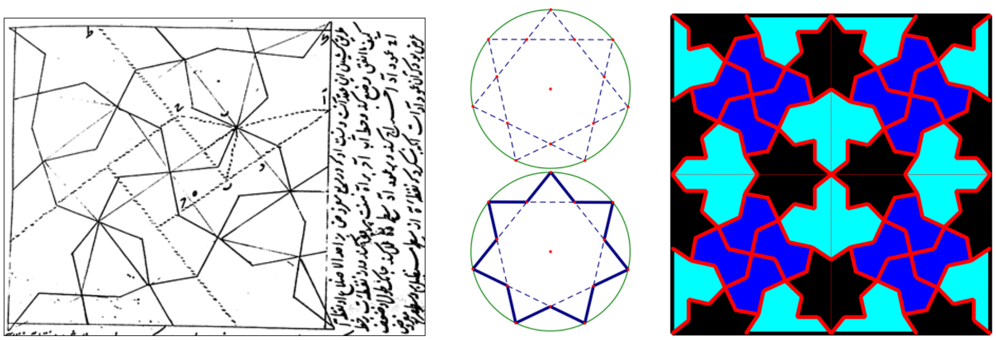

Figure 6: (L) *A girih in the treaties*; (M) *heptagram;* (R) *A heptagonal tiling for this girih*

Tiling with the heptagon and heptagram is rare (and therefore special) in tiling designs perhaps due to the fact that it is not constructible geometrically or because of the complexity involved in the tessellation with heptagonal or heptagram faces. The concave octagonal motif, which four of them make a shape with 4-fold rotational symmetry, is standard in Persian mosaic design, and can tessellate a plane by itself.

The treatise does not give a mathematical solution or approximation for composition of this girih. However, it suggests that by trial and error, one can create an acceptable girih solution. The following images in Figure 6, from left to right, demonstrate a suggested process.

Suggested solution by this paper: In a given square make an arbitrary small semicircle, with center *A*, that lies on one side of the square and intersects one corner of the square (see the left image in Figure 7). Inscribe one half of a heptagon inside of it, as is illustrated in the left image, and repeat this process for the other sides of the square. Connect *A* to *B*, the vertex of the heptagon, and extend the segment to meet a side of the original square. Repeat this process for all other sides of the square to create the square *CDEF*. The square *GHIJ* is constructed in a way that $GH = 2\ CD$. Extend the sides of this square and also divide the square *CDEF* into four smaller squares to create a square grid (the dashed mesh). Point *M* is the midpoint of *CF* and point *N* is the midpoint of *FM*. Point *O* is the center of the square. Construct *NO* and its reflection under the reflection line of line *CF*. Using *EF* as the reflection line, reflect *NO* and its reflection. Continue this process for other sides of the square to create four congruent concave octagons (called *Tabl*) that are attached in a rotational format. If the original semicircle is not sufficiently large, then there will be a gap between the sides of the generated heptagons and octagons, as is presented in the first image. By trial and error, one can increase the size of the radius of the semicircles that will increase the size of the squares as well, to the point that the sides of the two polygons coincide (approximately), as is exhibited in the second image in Figure 7. It should be mentioned that a complete coincidence is impossible because of the fact that the sizes of the angles of the two polygons of the heptagons and tabl are not exactly the same, despite their presentations in the second image. The interior angle of the tabl is 128.03° and the exterior angle of the heptagon is 128.57°. Even thought the angles are slightly different from each other they are proper approximations and in the final composition, one can erase one of the two sides that are very close to each other. By adding four segments around, as is shown in the right image in Figure 7 that exist in the original girih, the girih construction is complete. However the middle image in Figure 7, without those extra segments, which creates a more compelling tiling pattern, was used for constructing the right image in Figure 6.

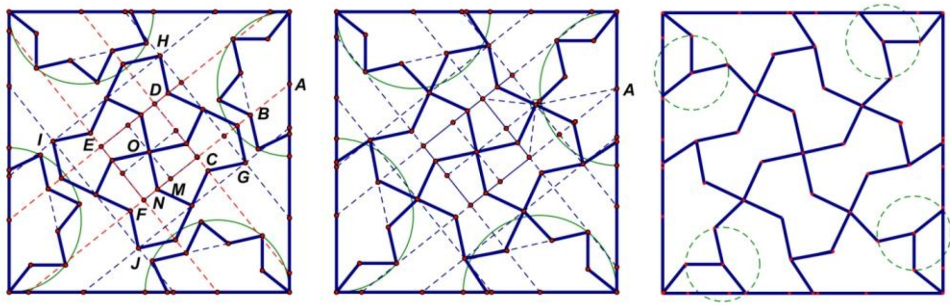

Figure 7: *The process of creating the girih by trial and error.*

3.2. A heptagonal tiling suggested by the treatise. The left image in Figure 8 is another attractive girih in the treatise, which is related to the heptagon. Suppose $\angle A$ is a right angle and B is on one of its sides. Divide this angle into seven congruent angles using a ruler and make angle $\angle BAG$, which is 3/7 of $\angle A$, where $BG \perp AB$. Find D the midpoint of AG. Find H in such a way that BH is congruent to AD. Make a parallel to AG from H to find R on AB. Construct T, the midpoint of RH. Find Z, the midpoint of TH. Construct TI, where TI is congruent to TZ and is parallel to BG. Find K, the intersection of AB and line HI. Find KL parallel to IT where L lies on RH. Now find F on a circle with center R and radius RK, where K and F are reflections of each other with respect to RH. Find C, the intersection of line FL and the other side of the right angle A. Point C is the center of a regular heptagon. Now complete the lower part of the girih according to the second image in Figure 8. The third image has been created from the second image using a 180° rotation about O. Some minor revisions are necessary to complete this girih and use it for a heptagonal tiling as in the right image in Figure 8.

Note that even though the construction process results in a heptagonal mosaic pattern, there is no explanation in the treatise to justify the steps in its construction.

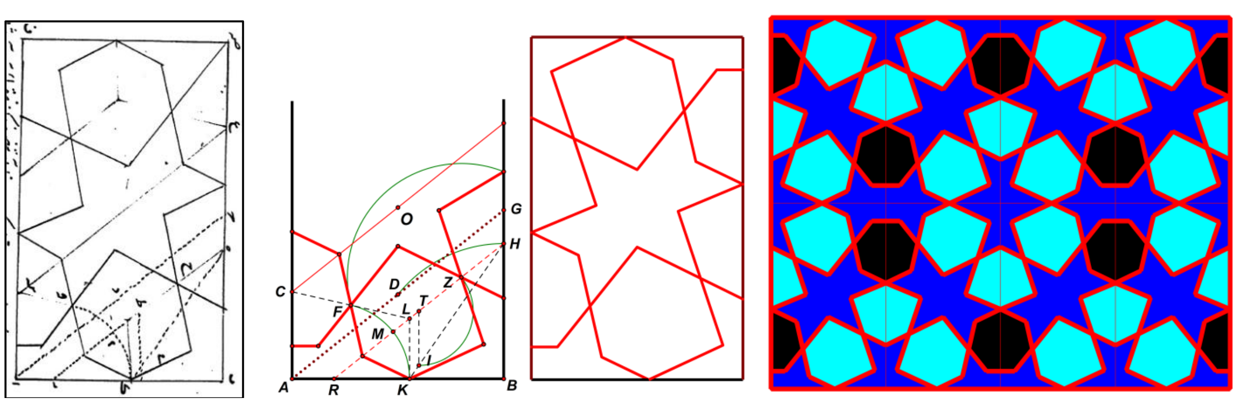

Figure 8: *The process of creating a heptagonal girih*

3.3. A torange girih. The left image in Figure 9 demonstrates a *torange* girih in the treatise. Torange, in Farsi means kite. Quadrilateral $DPKS$ is a torange with two right angles. So in this girih there are four toranges that exhibit a four-fold rotational symmetry design.

The following is a translation, with some modifications, of the composition process that is offered by the treatise: The square *ABCD* in the middle image in Figure 9, is given. Make a quarter circle with center *B* and radius *AB* to intersect the semicircle with diameter *AD* at *E*. Make a perpendicular from *E* to *AD* to intersect this segment at *F*. Select *P* on *AD* in a way that *PD* is congruent to *EF*. Using distance *PD* find points *Q*, *R*, and *S* on *AB*, *BC*, and *CD* respectively to construct an inscribed square inside of the given one. Now make a circle with center *P* and radius *PD* to interest *AD* at *T*. From *T* make a perpendicular to *PQ* to locate *K*, the image of *T* under the line of reflection *PQ*. Now rotate *PQ* and point *K* about center *O* using four-fold symmetry to create four right triangles (one of them is ΔPQL), and the square *KLMN* in the middle of the triangles. The girih is completed.

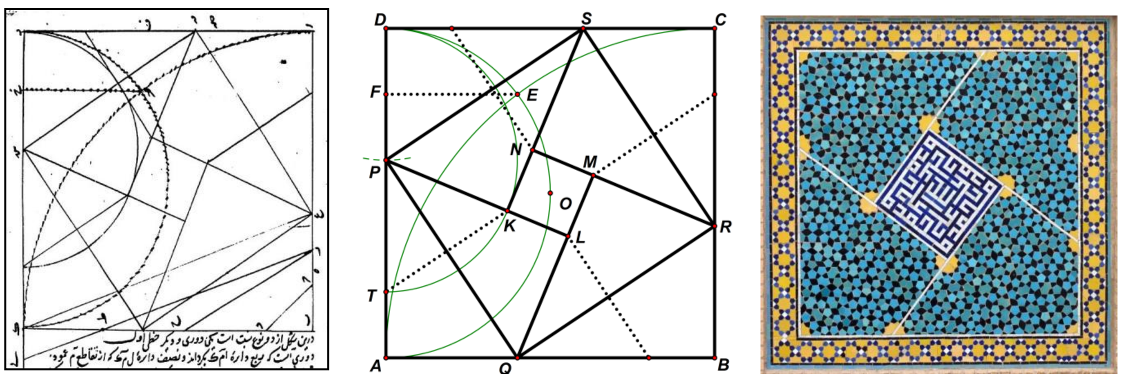

Figure 9: *Constructing a torange girih, and a ceramic design on a wall of a building in Isfahan, Iran.*

It seems that finding *P* the way that is described above is not necessary for the construction of this girih. Any arbitrary point *P* on *AD* results in a similar girih but with different angle measures for ΔPAQ and others. So there are infinitely many solutions for this girih if we relax the aforementioned condition for constructing *P*. To discover the reason behind the special choice of the treatise's author is left for the reader. An application of this girih with a different choice of *P* can be found as a mosaic design on a wall of *Chahar Bagh School* in Isfahan, Iran, which is presented in the right image in Figure 9.

3.4. A decorated torange girih. The left image in Figure 10 presents a girih that is constructed based on a torange girih as in 3.3. The difference is the right triangle that constitutes the entire design is a special one with a 60° angle. Unfortunately the explanation recorded in the treatise for this girih cannot be understood and the illustration cannot help either to find the way that the author constructed this special triangle. However, from the image of the girih it seems that the author's goal was to construct a decorated torange tiling using a regular hexagram.

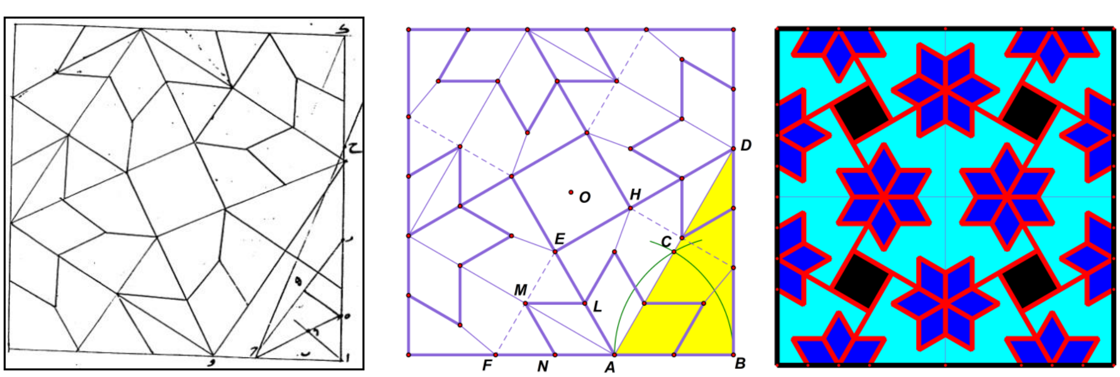

Figure 10: *Constructing a torange girih and its tiling composed using reflections*

The author was unable to follow the treatise's instruction for this girih. Therefore, the following approach is suggested: Segment *AB* is given. Construct the equilateral triangle Δ*ABC*. Find *D*, the intersection of line *AC* with the line that is perpendicular to *AB* from *B*. The angles of the right triangle Δ*ABD* are 30° and 60°. Then find the torange *ABDE*. Find *EH* with measure *BD* – *AB*. Construct the square with base *EH* and find *O*, the center of the girih. Rotating *ABDE* about this center creates the girih, as is demonstrated in middle image in Figure 10. From this image it is easy to understand the process of constructing the rhombus *NALM*, which is based on the construction of *M* on *EF*, as is illustrated in the figure (*EF*⊥ *AM*). This rhombus is used to decorate the girih as is shown in the middle image. The right image in Figure 10 is a tiling that is formed by this girih. From the construction, one may realize that the size of the rhombi and consequently the size of the hexagram may vary if the condition for finding *M* is relaxed.

3.5. A complex girih in the treatise. The left image in Figure 11 is another girih that is recorded in the treatise. Except for the great hexagram (the 12 sided equilateral polygon in black shown on the right) that is created using 12/3 star polygon, and *shesh tond* (smaller hexagon in dark blue) the tiling for this girih includes motifs that are not usual.

Most of the steps involved for the composition of this girih are not clear. Moreover, from the beginning there are assumptions that are not sufficient to create this girih, as is presented in the middle image of Figure 12. The instruction recorded in the book is as follows:

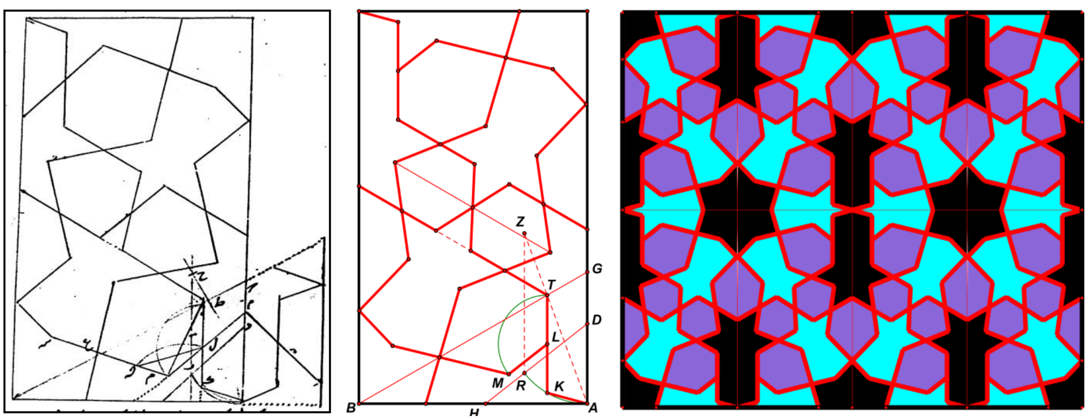

Figure 11: *Constructing a torange girih and its tiling*

Instruction offered by the treatise: Make a right triangle (with angle 30°) Δ*ABG*. Consider arbitrary points *D* and *H* on this triangle as in the middle image and make segment *DH*. Find *R* on *DH* in a way that *DR* is congruent to *DA*. From *R* make a perpendicular to *AB* and find point *Z* on this line so that *RZ* = 2 *AR*. The intersection of *AZ* and *BG* is *T*. From *T* make a parallel to *AG* to intersect the arc *AR* with center *D* on point *K*. Find *L* the midpoint of *TK*. Construct *LM* congruent to *LK* in a way that *ML* and *HD* are parallel. Construct an angle congruent to ∠*MLT* with center *M* and one side *ML*. Now complete the process!

It is obvious that assumptions, such as the four arbitrary points in the beginning and then leaving the problem incomplete, may result in many shapes, none of which can create the girih. Using a dynamic geometry system, one may find an acceptable solution by moving points *A*, *B*, *G*, *D*, and *H*. There are many modifications and assumptions that are needed to create the middle image in Figure 12. It is possible that the treatise's author copied the design from another source and then tried to justify the steps in the construction. In fact, adding a part of the reflection of the girih in the left image may be a reason that the author was in search of finding a unique solution based on specific locations for the aforementioned four points.

3.6. Making a girih for a new 10-12 Petal roses tiling. Figure 12 demonstrates a girih and the generated tiling that was studied in a previous paper in the 2015 proceedings [3]. The girih on the left shows a design that is constructed as a combination of two different roses of 10 and 12 petals girihs in the two middle images. As is evident from the left image, the treatise's author has tried to analyze the design and provide instruction for its construction that seems is not correct. Nevertheless, from the rays emanating from the bottom-left and top-right of this girih, a person who is familiar with the radial grid approach can find a way to compose this girih and create the tiling pattern on the right.

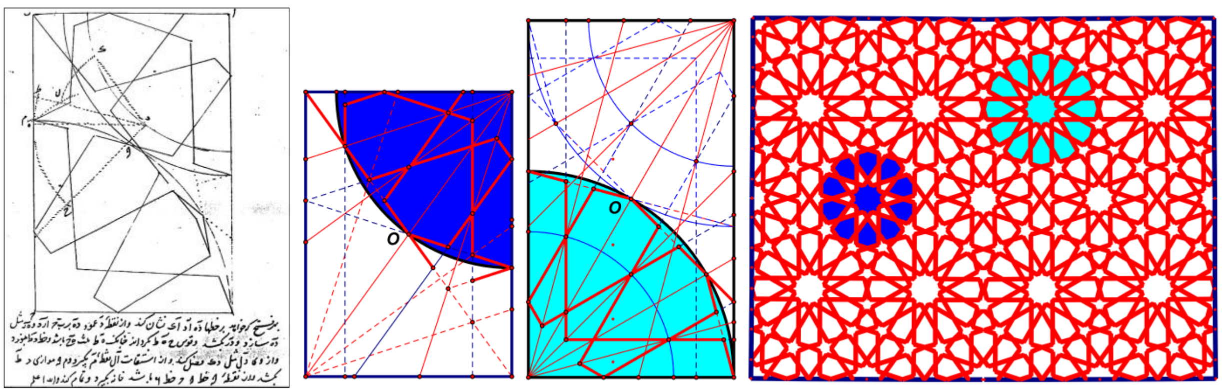

Figure 12: *A girih, a 10-petal girih, a 12-petal girih, and the tiling made from their combinations*

The treatise includes another 10-12 petal rose girih, as is shown in the left image of Figure 13. For this girih there is not any instruction or any auxiliary lines and curves, similar to the aforementioned girih, to help create a method to compose this new one. The main difference between the two 10-12 rose girihs in Figures 13 and 14 is that the positions of the roses in the girihs are different. The 12-petal rose girih in Figure 13 includes three complete petals. But the new girih in Figure 14 includes two complete and two half petals. Moreover, the position of the 10-petal rose is now different from its original position as in Figure 12. The right image in Figure 13 exhibits the new 10-12 petal rose tiling.

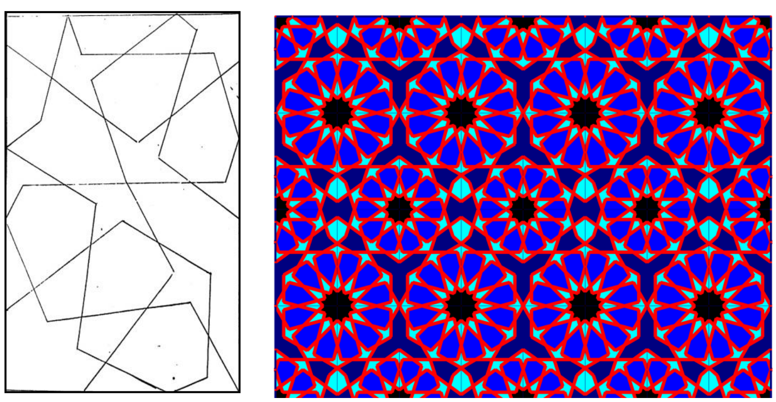

Figure 13: *A girih in the Interlock treatise and is tilling*

For this girih, previous girihs were used to arrive at the solution (sse Figure 12). To construct the desired 10-petal girih in Figure 13, it is enough to consider the girih in Figure 12 and apply a reflection and a rotation. However, for the new 12-petal rose girih, one should consider the complete generated 12-petal rose in the previous problem and then select~~s~~ one fourth of it in a way that includes two full and two half petals.

The two left images in Figure 14 show the two new girihs for this problem. Unfortunately, there is not a convenient method, as in the previous case, to coincide the two centers and complete the process. Here, the two points of *A* and *M* that are supposed to coincide are not the center points. One notes that in order to compose a new combined girih it is necssary that the right side of the 10-petal rose girih coincides with the right side of the other girih. If one employs the same proportions as the previous problem (where $ST = OP$), then there will be a gap as can be seen in the third image from the left in Figuare 14. To avoid this problem, after the first construction of the 10-petal rose girih, the measure of *AB* should be obtained. For a perfect match on the right sides of the two girihs where *A* and *M* coincide, the size of each side of the 12-petal rose girih, which is a square, should be twice *AB* ($CD = 2\ AB$). It is easy to show that *M* is the midpoint of *CD*. For this, note that $\angle COM = \pi/6$ so $CM = 1/2\ MO$. But then $OM = OP$, as both are located on a circle with raduis *OP*. Therefore, $OP = 2\ AB$. Hence after dividing the right angle $\angle R$ into five congruent angles, one should locate an arbitrary point *T* on the third ray emanating from *R* and consruct the rectancular 10-petal rose girih in the first image on the left. After that, one should construct a square with sides congruent to $2AB = OP$. Using this square for its frame, it is not difficult to construct a 12-petal rose girih as shown in the second image in Figure 14. While the third image shows an incorrect combination, the fourth image exhibits the new 10-12 petal rose girih.

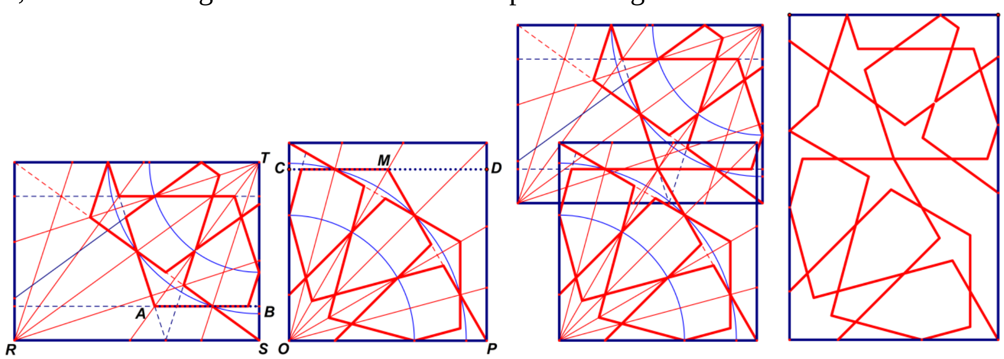

Figure 14: *The process of composing a new 10-12 petal rose girih*

4. Conclusion

The treatise *Interlocks of Similar or Complementary Figures,* includes many interesting dissection puzzles and girihs that are useful in education for creativity and learning purposes. The document reveals that the author was a mathematician, with a background in the mathematical properties of polygons in order to make the puzzles. He was also familiar with the process of composing some captivating tiling designs. The instructions for the construction of many puzzles and girihs in this treatise are not clear and comprehensive. Nevertheless, it is the only document from the past that presents instructions for girih compositions. Another conclusion is that during the time that this treatise was written, a few mathematicians were in deep contact with tiling designers and artists. This relationship, as is evident in the treatise, was solely based on the knowledge of mathematics. In spite of the fact that the treatise is old, it provides many functional ideas that should be included in current geometry curricula or in the architectural design workshops.

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