Helical Petrie Polygons

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Abstract

A Petrie polygon of a polyhedron is defined as a circuit of edges such that exactly two consecutive edges lie in the same face. In the case of an infinite polyhedron (or honeycomb or sponge) this definition leads to an infinite helical path, rather than a circuit. Since there is a one-to-one correspondence between the edges of a polyhedron and its dual it is possible to transform the Petrie polygons of one into the other, and if the transformation is continuous there can be interesting intermediate configurations. If the sponge is regular the Petrie polygons can be replaced by circular helices passing through polyhedral vertices. The helices can be transformed in an analogous way to the polygons, again with interesting intermediates.

Three dimensional networks can be considered as the edges of a polyhedral space packing, although the faces of the polyhedra may not always be planar. Often a sponge can be made by removing some of the faces, and such cases are briefly considered. Arrangements of circular helices previously used by artists and researchers are recalled, and the possibility of finding further arrangements derived from 3-D networks is suggested.

Petrie Polygons

While still at school (with H.M.S.Coxeter) John Flinders Petrie first considered the non-planar zigzag equatorial polygons found in regular polyhedra [1]. A more general definition that can be used for any polyhedron specifies a *Petrie polygon* as a path (circuit if the polyhedron is finite) of edges such that any two (but no more) consecutive edges lie in the same face (figure 1). Since any edge is shared by exactly two faces, there are two possible paths once the first edge is chosen, but after that the path is uniquely determined. For the same reason every edge lies in exactly two Petrie polygons.

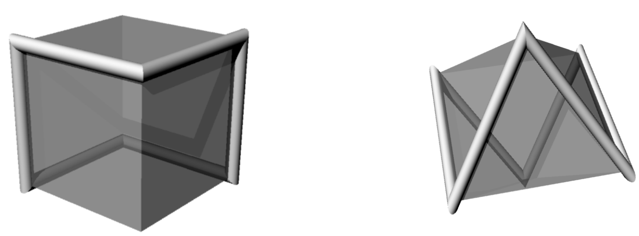

Figure 1: *Petrie polygons of the cube and octahedron.*

Two polyhedra are *dual* if there is a one-to-one correspondence between the faces/vertices of one and the vertices/faces of the other, and it follows that there is a one-to-one correspondence between the edges of dual polyhedra since each edge lies both between a pair of vertices and between a pair of faces. In particular there is a one-to-one correspondence between the Petrie polygons of dual polyhedra, since if one edge in a circuit is incident with a vertex so is the next edge (but there are no others), exactly matching the relationship between edges and faces in the definition of Petrie polygons. Although there can be problems in determining specific well-defined duals of polyhedra in general, there is no difficulty if it is regular, and it is possible to smoothly transform the Petrie polygons of a polyhedron to those of its dual [2]. Figure 1 shows a cube with one of its Petrie polygons and its dual, the regular octahedron, with the corresponding Petrie polygon.

Infinite Polyhedra (Sponges)

Petrie also discovered two regular infinite polyhedra, and Coxeter found a third [3]. Petrie's form a dual pair, one, {4,6}, with six squares at each vertex, the other, {6,4} with four hexagons at each vertex. Coxeter's, {6,6}, with six hexagons at each vertex, is self-dual. They can be constructed from packings of polyhedra by replacing pairs of coincident faces with a single face, and removing some of them: Petrie's from a packing of cubes, removing half the squares, and from a packing of truncated octahedra, removing all the squares; Coxeter's from a packing of tetrahedra and truncated tetrahedra, removing the triangles (see figure 3). These structures are known variously as the regular sponges, regular honeycombs, regular skew polyhedra or regular skew apeirohedra.

The faces around a vertex form a zigzag circuit, rather like Petrie's polygons, so that, unlike normal polyhedra, there is no choice of relative scale in a compound of a regular sponge and its dual: the vertices of one must lie at the centres of the faces of the other.

There are many more, less regular, sponges: some derived from polyhedral space packings (for example by removing half the hexagons in a packing of truncated octahedra, rather than squares as in the regular sponge), and others made by constructing tunnels between polyhedra (for example exploding a packing of truncated octahedra, filling the gaps with hexagonal prisms and removing all the hexagons produces a sponge with five squares at each vertex) [4].

If non-planar faces are allowed (as in so-called saddle polyhedra [5]) then there are further possibilities [6]. For example the first stellation of the rhombic dodecahedron (sometimes called Escher's solid) will fill space. It can be considered as an octahedron with faces that are skew hexagons with 90° angles, which could be filled with a saddle surface. Proceeding as before and removing half the faces leaves (if the area of the saddle surface is minimised) what Pearce [6] calls a labyrinth, and is an example of a triply periodic minimal surface [7]. This one is Schwartz's D surface (figure 2).

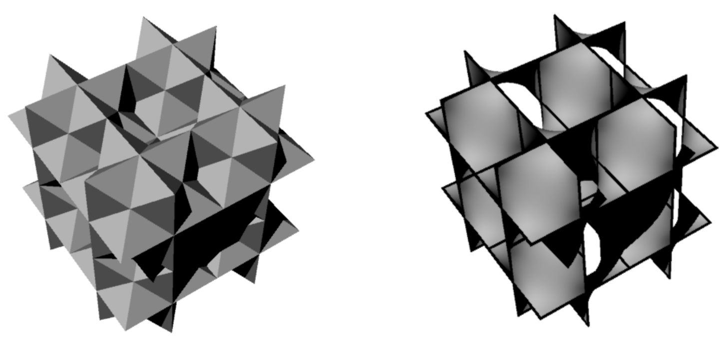

Figure 2: *A packing of stellated rhombic dodecahedra and Schwarz's D surface.*

Petrie Polygons of Sponges

If Petrie polygons are constructed according to the standard definition, the regular sponges yield helical paths (figure 3). They are helical in the sense that they have screw symmetry, and rather than polygons they are more accurately termed helical skew apeirogons. They will be either right-handed or left-handed depending on the choice made for the second edge when applying the definition.

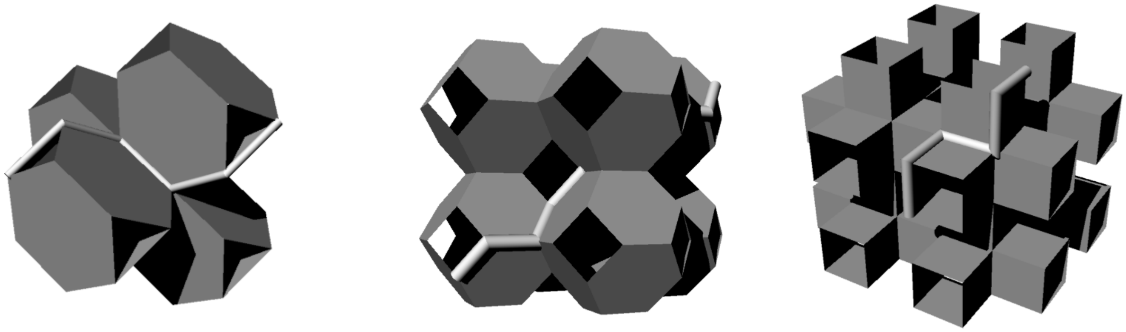

Figure 3: *Petrie polygons of the three regular sponges.*

Less regular sponges also have helical apeirogons as Petrie polygons, although obviously if the faces include more than one type of polygon, the apeirogons are not regular, and if there are different dihedral angles there will again be more than one size of angle between the edges of the apeirogon.

Things are different if the faces are non-planar. For example in Schwarz's D surface (figure 2) the Petrie polygons are not helical but correspond with skew hexagons that are edges of the faces that have been removed from the packing of saddle polyhedra (figure 4).

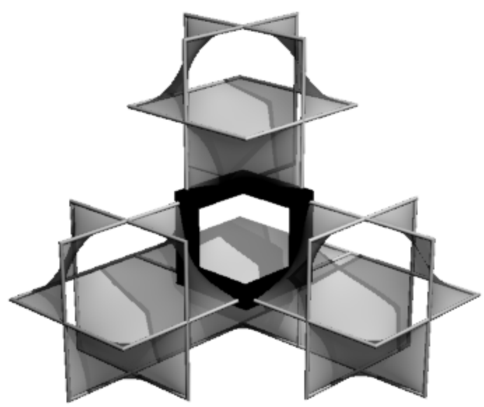

Figure 4: *A Petrie polygon (black) of Schwarz's D surface considered as a regular sponge.*

Transforming the Petrie Polygons. Just as with standard (spherical) polyhedra it is possible to smoothly transform the Petrie polygons of a sponge into those of its dual. In this case an intermediate stage consists of just a line along the axis of screw symmetry. The transformation can be extended, typically by varying a parameter, and it may take other values when vertices or even edges of the Petrie polygons coincide. It is generally not easy to see how such configurations relate to known three dimensional polyhedral packings (if they do at all) but figure 5 shows an example where the vertices of the Petrie polygons of the regular sponge {6,4} have been moved further from the axis of symmetry until the edges meet in groups of three and four at their mid-points. They lie along edges of the packing of stellated rhombic dodecahedra. Notice that the edges of the transformed Petrie polygons do not lie along the skew hexagons that are retained in the saddle octahedron (see figure 2).

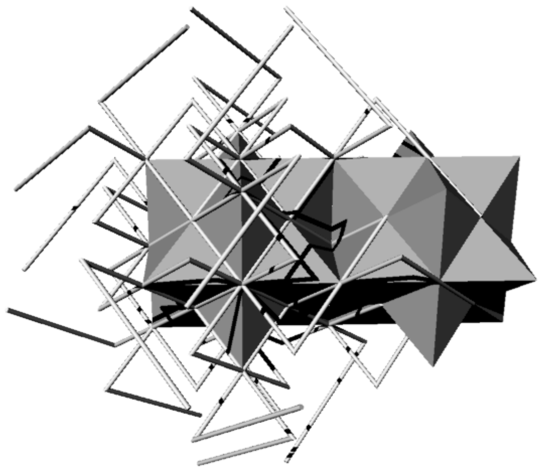

Figure 5: *Transformed helical Petrie polygons that lie along edges of stellated rhombic dodecahedra.*

Using Circular Helices

In the particular cases of regular sponges it is possible to construct circular helices passing through the vertices of the Petrie polygons, corresponding to the edges. Since every edge of a polyhedron lies on two Petrie polygons choosing either right-handed or left-handed helices will provide a match for all of the edges, and generate attractive representations of the regular sponges without mirror symmetry (figure 6).

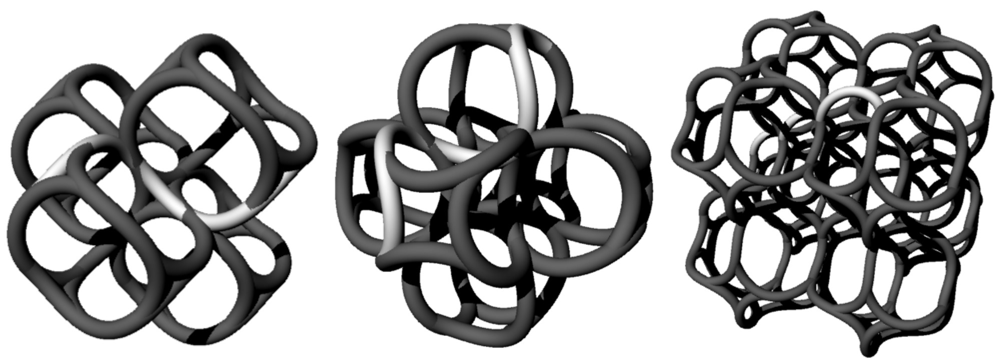

Figure 6: *Representations of the regular sponges constructed with circular helices.*

Figure 7: *Elements, from the regular sponges represented by helices, with the faces filled in.*

Filling in the surfaces in a way analogous to the construction of saddle polyhedra generates polyhedral objects that could be used to make labyrinths (figure 7). Notice the edges of filled-in faces are convex and those of the missing faces are concave.

Transformations. The circular helices can be transformed in an analogous way to the apeirogons simply by varying the radius. Clearly when the radius is zero the helices become lines that lie along the axes of screw symmetry of the sponge. Increasing the radius beyond the value that corresponds with a regular sponge shrinks the hole (as well as changing the alignment, so the helices no longer touch) until eventually the helices intersect. In the case of {6,4} it is a square that shrinks, so that four helices intersect lying approximately along a pair of perpendicular lines (figure 9). {6,6} has tetrahedral gaps, so six helices intersect when it has shrunk completely (one for each edge of a tetrahedron), and {4,6} has cubic gaps so there is an intersection of twelve helices. In both cases a complicated network is produced.

Increasing the radius still further produces helices that wind around each other in the way that has been used extensively by Alexandru Usineviciu [8, 9], and he has already created a sculpture of a tetrahedron that is a fragment of the expanded helices of{6,6} (figure 8) [private correspondence].

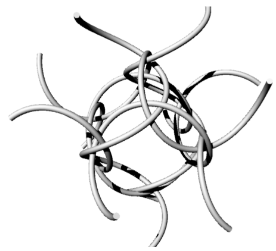

Figure 8: *Interlaced helices forming a tetrahedral arrangement.*

If the size of the radii of the helices of {6,4} is taken as 1 then as it is reduced to zero the helices become straight lines. The dual, {4,6}, reached by going beyond zero occurs when the helices have radii that are twice the size, but measured in the opposite direction, so it can be considered to be -2. When the radii are twice the size the helices intersect in fours (the arrangement already mentioned), and when the radii are quadrupled the helices correspond with the vertices of the apeirogons in figure 5. Of course they cannot lie along the edges of the stellated rhombic dodecahedron, which are straight lines, and the configurations match only at the 6-fold vertices (figure 9).

Figure 9: *Doubling and quadrupling the radius of the helices of {6,4}.*

When the radii of the Petrie helices are doubled, and helices intersect in fours, other intersections occur that do not correspond with squares in {6,4}, so that all of the vertices of the Schwarz D surface are represented. The arcs of helices (with a radius of 2) shown in the figure correspond with one turn of the helices of the dual cubic sponge with a radius of -2 (changing the sign of the radius, equivalent to a half-turn about the screw axis, changes the orientations, and the helices of {4,6} have the same size radius). It is not possible to remove half of the helices to leave a helical representation of the 4-valent 3-D network, as careful inspection of the helical arcs corresponding to the edges of a skew hexagon will make clear.

Some Other Arrangements of Helices

Helices derived from the Petrie polygons of the regular sponges and their transformations by no means exhaust the possible ways of arranging helices. Reference has already been made to the work of Alexandru Usineviciu, introduced to Bridges by Paul Tucker, who has made his own exploration of "moorish fretwork" [10, 11], and Koos and Tom Verhoef's zigzagzeg and zigzagzegzug mitred constructions [12] include some that are more than interlacings of parallel arrangements of helices.

The 3-D networks first investigated by A.F.Wells (see [5] for extensive references) are the basis of Pearce's work on labyrinths, but their use to find arrangements of helices (other than Wells's (10,3)a, in which the helices are immediately obvious [13]) has hardly been considered. It seems that much remains to be discovered.

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