A Divine Error

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Abstract

In the 'De Divina Proportione', illustrated by Leonardo da Vinci, author Luca Pacioli formulated a brash statement about a theorem claiming six summits of pyramids of the elevated icosidodecahedron lay in one plane. Pacioli provided no proof while Leonardo's illustration for this statement contains errors. Thus, Jos Janssen (The Netherlands) suspected Pacioli's theorem was not true, and Rinus Roelofs (The Netherlands) confirmed his doubts using geometric computer software. Here, we confirm their rejection using arithmetical techniques Pacioli could have employed in his time. And yet, the abundant literature about 'De Divina Proportione' never revealed this error in the past 500 years.

Introduction

Once upon a time, a popular science magazine in Dutch, called EOS, proposed a paper to Scientific American, as usual, because of their close collaboration (see [2]). The topic seemed but a regional news item, since it was about a Dutchman's discovery, Rinus Roelofs, of an error by Leonardo da Vinci. Yet, the translation caused unexpected controversy, due to the numerous believers in the divine status of Leonardo. They went at length trying to excuse him, objecting the errors were matters of interpretation or inaccuracy, or not even mistakes at all but 'mysterious messages'. This unexpected hostility became a motivation for continuing publishing and talking about the subject, though, admittedly, the topic is not of great importance, as everyone makes errors, and slip-ups only add to the genius of Leonardo. It was also a motivation to have a closer look on Leonardo's work, and it turned out Leonardo made many mathematical errors (see [3], [4], [5], [6]), but their publication was only accepted by diplomatically smoothing them down as 'inaccuracies in the drawings'. Our defensiveness made chemist Jos Janssen (Leiderdorp, The Netherlands) stumble on yet another error implicating Leonardo da Vinci, the one that will be discussed in the present paper. It was confirmed by mathematical artist Rinus Roelofs (Hengelo, The Netherlands) using a geometric computer software, but his drawing did not convince the believers. A first version of this current paper (see [7]) was commented by some readers who simply said they did not see the error, and they demanded a traditional computational proof (which we will provide here). The Leonardo adepts also asked to quote the extensive literature and the many specialists who already wrote about the topic (see [10] or [11]) – but why? The error is undeniable!

This time the slip-up is not directly Leonardo's, but Luca Pacioli's, for whom Leonardo made the illustration for the book 'De Divina Proportione', or 'On the Divine Ratio'. The name of Franciscan Fra Luca Bartolomeo de Pacioli (1445-1517) is sometimes written as Paciolo or Paccioli because Italian was not a uniform language in his days, when, moreover, Italy was not a country yet. He can be called a Tuscan, because of his birthplace of Borgo San Sepolcro, but he also studied in Venice and Rome, and spent much of his life in Perugia and Milan. In service of Duke and patron Ludovico Sforza, he would write his masterpiece, in 1497 (although it seems more correct to say the work was written between 1496 and 1498, because it contains several parts). It was not his first opus, because in 1494 his 'Summa de arithmetic, geometrica, proportioni et proportionalita' had appeared; the 'Summa' and 'Divina' were not his only books, but surely his most famous ones (see Fig. 1a).

For hundreds of years the books were among the most widely read mathematical bestsellers, their fame being only surpassed by the 'Elements' of Euclid. The 'Summa' was more popular than the 'Divina', though this no longer applies today because we now are especially impressed by the illustrations Leonardo da Vinci made for this book. They were the first truly insightful spatial representations of polyhedra, similar to contemporary 3D computer drawings. Perhaps this collaboration between Pacioli and Leonardo was glorified in a well-known portrait from 1495. It is usually attributed to Jacopo de' Barbari but this statement is questionable, as is the suggestion Leonardo (or Dürer?) was the person painted in the background (see Fig. 1b).

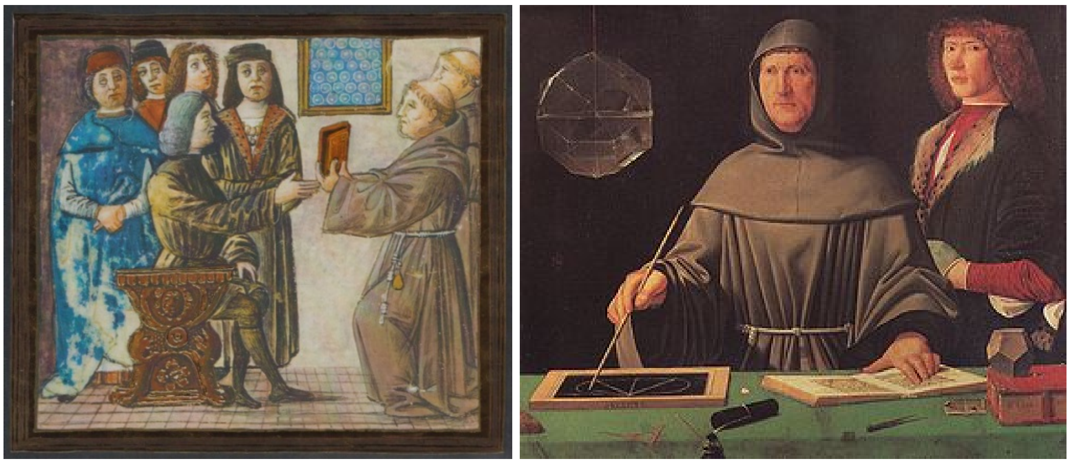

Figure 1: *Portraits of Luca Pacioli: one from the introduction of his book, where he donates it to Duke Ludovico Sforza (left) and one from an overly well-known painting (right).*

Because of financial difficulties, 'The Divina Proportione' was only printed in 1509, in Venice, when Paganinus de Paganinus decided to take up the challenge of editing a mathematical book. Thus, it was more than ten years after the publication of the manuscript, of which two versions remain today, one in Geneva and one in Milan (see [1]). In the latter two, the artwork is hand-painted while in the printed version they were replaced by woodcuts, with different degrees of precision and accuracy.

An incorrect statement

In section 'LII' of 'The Divina Proportione' Pacioli describes how pyramids with identical edges are built on the faces of a solid he called 'duodecedron abscisus', but that we call icosidodecahedron today (see Fig. 2). The following translation is based on the French translation (see [8]); it does not always concur with the Spanish and German version (see [9]), but the essential parts are identical:

> *And the body that is created in this way, is composed by the flat truncated dodecahedron [icosidodecahedron], inside, which shows itself to the mind only through the imagination, and by 32 pyramids, of which 12 are pentagonal, all of equal height, and of which the 20 others are triangular, all of equal height. The bases of the pyramids are the faces of the aforementioned dodecahedron [icosidodecahedron] and they mutually correspond, that is to say, the triangles to the triangular pyramids and the pentagons to the pentagonal pyramids. Placed on a plane, this body will always rest on 6 tops of pyramids, one of them being a pentagonal pyramid, the other five triangular. When*

this body is seen in the air, it seems at first sight astounding that these vertices satisfy this property, but something like this, noble Duke, is of such great abstractness and deep science that I know that who understands me will not deny it. As for the dimensions of this body, they are obtained by the very subtle practice of primarily algebra or almucabala, which is known to few people, and is well demonstrated by us in our work, with methods allowing understanding them easily.

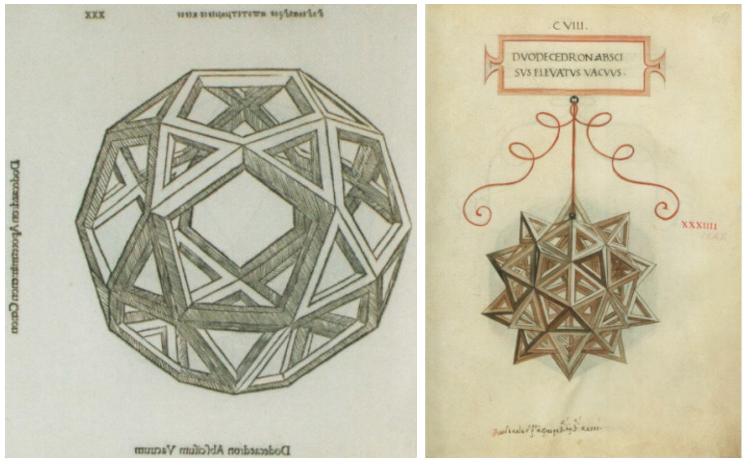

Figure 2: *The icosidodecahedron in the printed version of 'De Divina Proportione' and the elevated icosidodecahedron from the Milan version.*

Addressing the 'noble Duke' in a mathematical text was not unusual in those times. After all, Ludovico Sforza was the patron who had paid Pacioli. The self-flattering wording about the use of 'algebra and almucabala' seems outdated too, though the author was right when he pretended to be checking geometric properties through calculations that were little known in his time – after all, Descartes was just born. Now at first glance it seems that the six points mentioned by Pacioli indeed lie in one plane. When building some real models of the elevated icosidodecahedron, the conclusion is not persuasive. Of course Zometool would be ideal, but Paul Hildebrandt informed us it does not allow the construction of a model in which all edges are exactly equal (see fig. 3a). Using the 'Tubespace' connectors and tubes, the 3D model wobbles slightly, and that would show the 6 points do *not* lie in one plane, but this only happens when the model is stuck firmly together (see fig 3b). A PolyPuzzle model does not allow a very precise assembly as paper willingly folds in any direction (see fig 3c). And so, no decisive conclusion about the truth of Pacioli's statement can be drawn based on those models.

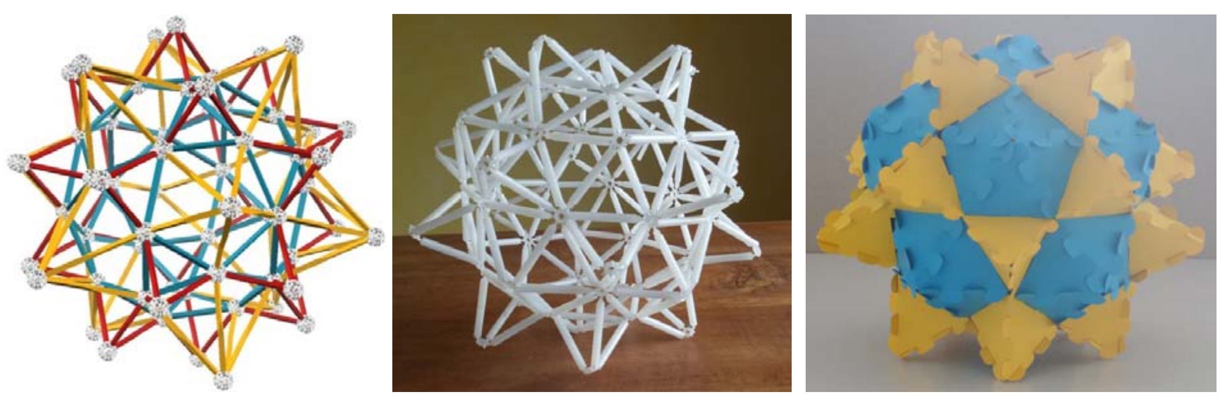

Figure 3: *Models in Zometool (left; image P. Hildebrandt), Tubespace (middle) and PolyPuzzle (right).*

However, Jos Janssen started to doubt when he saw yet another error, on Leonardo's illustration for this theorem: some lines at the top of the figure are broken, while they should have been straight (see the dotted lines on Fig. 4a). And it surely aren't the only errors in this drawing: once the attention is drawn on this illustration, many an observer, such as Carlo Séquin (Berkeley), will quickly start to 'find the seven errors' (for answers: see Berkeley). Now Janssen corresponded about Pacioli's theorem with the author who on his turn asked Rinus Roelofs to run a quick check using his favorite 3D software Rhinoceros. It turned out the middle pyramid clearly goes through the plane formed by the five triangular pyramids surrounding it (see Fig 4b): Pacioli's statement is incorrect.

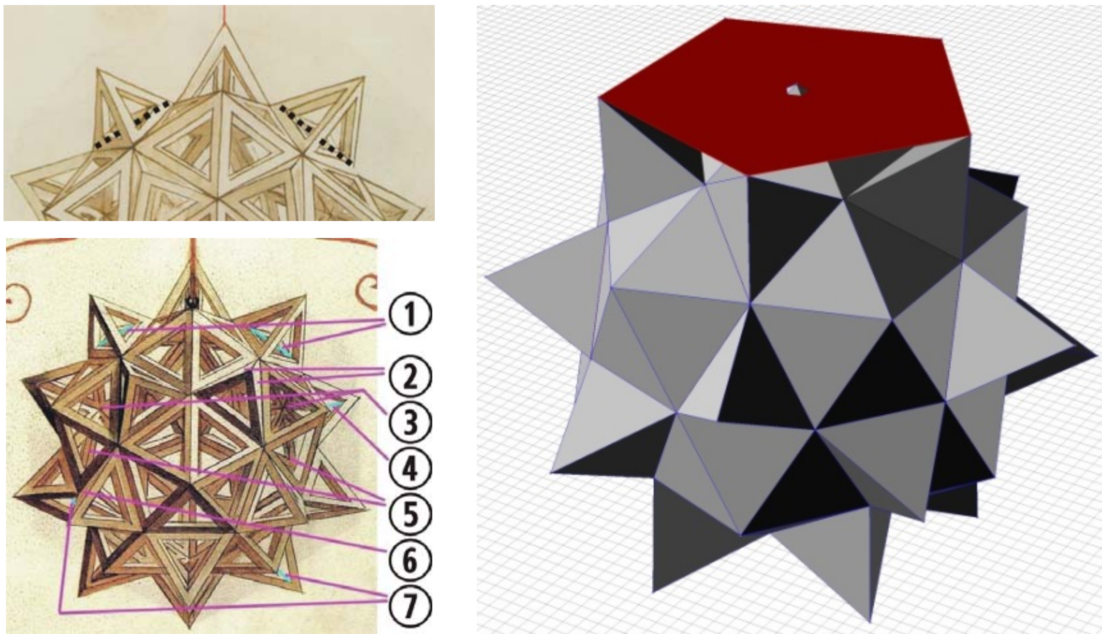

Figure 4: *Detail of the drawing in Fig. 2 (above left); Séquin's seven errors (below left); Roelofs' invalidation of Pacioli's theorem (right).*

The computational verification

The erroneous statement is printed in black and white, and thus is cannot be rejected as a matter of interpretation or as an inaccuracy. Of course, Roelofs used a modern computer software, and one can wonder if Pacioli could have proven his 'theorem' after all. We will show he could have, since we will provide a computation done in a way Pacioli could have done it, that is, without using trigonometric functions and using only calculations similar to those in his 'Divina Proportione'. We start by cutting the elevated icosidodecahedron in half, and compute the heights of the tops of the different kinds of pyramids (see Fig. 5).

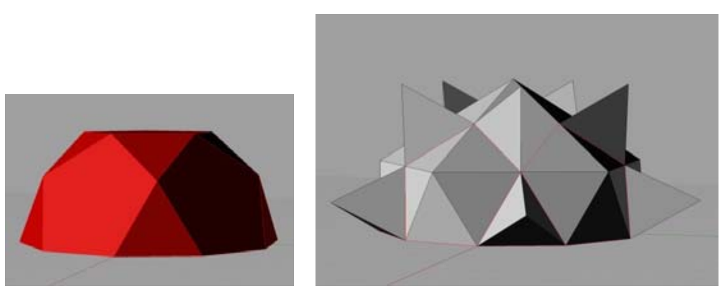

Figure 5: *Half an icosidodecahedron and half an elevated icosidodecahedron.*

We consider the two kinds of pyramids separately. First, there is the one in the middle, the pyramid with a pentagonal basis standing straight above the icosidodecahedron. Together with the other pyramids, they form exactly one half of a regular icosahedron, one of the Platonic solids studied by Pacioli. If the icosidodecahedron has side 1, this icosahedron will have side 2, and its radius is $\sqrt{\sqrt{5}(1+\sqrt{5})/2}$. Here, $\sqrt{(1+\sqrt{5})/2}$ is Pacioli's 'divine proportion', for which we avoid using φ as Pacioli never used this notation (it became 'fashionable' only about 350 years later). Thus, this is the height of the drawing in Fig. 6, left.

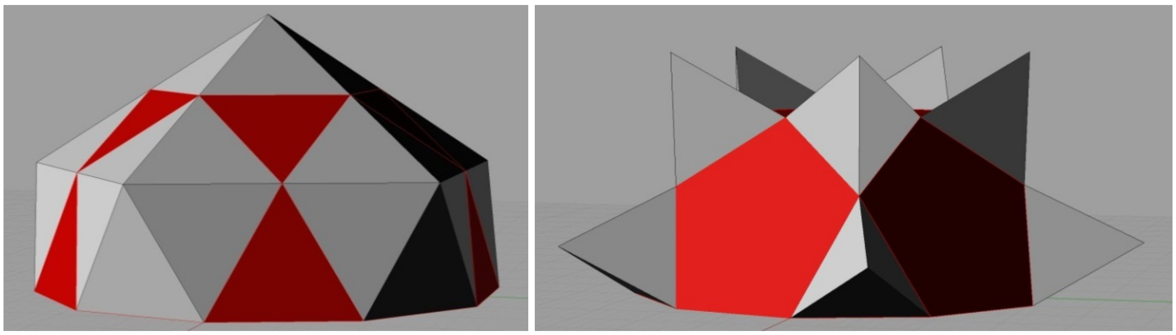

Figure 6: *Half a regular icosahedron leads to the height of the middle (left), but the surrounding tetrahedrons need more computational work.*

Now we compute the height of a top of a triangular prism with respect to the same plane going through the center of the polyhedron and perpendicular to the line connecting the center and the top of that icosahedron with side 2. The pyramid ABCQ is a regular tetrahedron with side 1 and thus height $\sqrt{\frac{2}{3}}$ (see fig. 7). The pyramid ABCO has an equilateral triangle ABC with side 1 as base and edges to the top O with lengths $\frac{1+\sqrt{5}}{2}$ since the radius of an icosidodecahedron with side 1 is $\frac{1+\sqrt{5}}{2}$. Its height is $\sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2-\frac{1}{3}}$. Thus, OQ = $\sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2-\frac{1}{3}}+\sqrt{\frac{2}{3}}$.

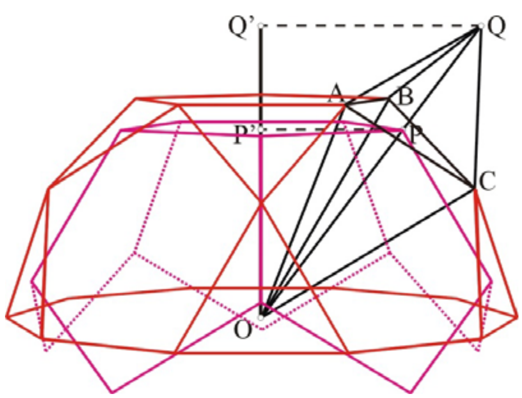

Figure 7: *The projections of the heights of two pyramids and the height of the top of the middle pyramid.*

To find the projection OQ’, we observe that $\frac{OQ'}{OQ} = \frac{OP'}{OP}$. Now Pacioli could have known OP’ is to OP as the radius of the sphere inscribed in a dodecahedron is to the radius of the sphere circumscribed to that dodecahedron:

$$\frac{OP'}{OP} = \frac{\frac{1}{2}\sqrt{\frac{5}{2}+\frac{11}{10}\sqrt{5}}}{\frac{\sqrt{3}}{4}\left(1+\sqrt{5}\right)}, \text{ and thus OQ'} = \left(\sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{3}} + \sqrt{\frac{2}{3}}\right)\frac{\frac{1}{2}\sqrt{\frac{5}{2}+\frac{11}{10}\sqrt{5}}}{\frac{\sqrt{3}}{4}\left(1+\sqrt{5}\right)}.$$

The expression for OQ’ is rather complicated when compared to the most involved expressions in ‘De Divina Proportione’. Pacioli could have suspected the expression for OQ’ would not simplify to √(√5(1+√5)/2), the height of the middle pentagonal pyramid on top of the icosidodecahedron, but perhaps he got stuck in his computations, as this easily happens when doing them by hand without using sine-cosine functions. It happened to the author too; it is tedious work.

Today, we can rather easily do a numerical check: √(√5(1+√5)/2) equals 1.90211…, while OQ’ = 1.84997… The error is minor indeed, and thus usual real 3D models will lack the precision to show this difference. Perhaps Pacioli had a simple wooden model that led him to announce his ‘theorem’ about the six coplanar points with so much commotion to Duke Ludovico Sforza even before getting the exact proof. Moreover, he knew he could do the computations, since they were within his reach, as shown above. And after all, isn’t making a bold announcement to please a sponsor an audacity that is still very much in use today?

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