# Double Strip Patterns: Between Strip Patterns and Wallpaper Patterns

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**Abstract**

Many examples of ethnographic art have designs which can be viewed as taking two copies of a strip pattern and sewing them together along one edge of the strips. In many cases, the resultant designs have symmetry that is not appropriately modeled by either traditional strip patterns or by wallpaper patterns. For example, while most of these double strip patterns can be extended to wallpaper patterns, that extension is not always unique and, we argue, is not necessarily a good representation of the artistic intent of the person who created the design. We classify the possible (one-color) double strip patterns and show artifact examples of many of them. In particular, we focus on a large number of such designs from pre-colonial Peru and from Papua New Guinea, demonstrate the substantial differences between which classes of double strip patterns these two regions use, and offer some suggestions as to the reasons behind these differences.

## 1. Introduction & Motivation

The "strip symmetry patterns" classify the designs that repeat a motif along a line, while the "wallpaper patterns" classify those that cover a full plane. The seven possible strip symmetry patterns can be described using the crystallographic notation *pabc*, where the "p" represents the *p*eriodic repetition, the "a" is either "m" if the design has a vertical *m*irror reflection, or a placeholder "1" if it does not; the "b" is either "m", "a", or "1" if the design has a horizontal reflection, horizontal glide reflection, or neither; and the "c" is either "2" if the design has a rotation of ½ or "1" if it does not (see [1] or [5] for details). Then the standard classification theorem says the collection of symmetries of the design must be one of the seven groups *p111*, *pm11, p1m1, p1a1, p112, pma2,* or *pmm2*. This classification usually does a good job of capturing, or modeling, the symmetries that an artist or craftsman puts into a (one-color) linear design. Consider, though, the design of figure 1, where we show: (a) an ethnographic example of a common design from Papua New Guinea; (b) an abstraction of that design; and (c) one-half of that design, which is also a strip pattern. In this figure we have colored half the motifs to demonstrate a problem with modeling this design repetition using strip symmetries: The strip of 1(c) has a glide reflection which maps white motifs to colored motifs; but the full design of 1(b) has *no* symmetry which maps white motifs to colored ones. The glide reflection of 1(c) fails to be a symmetry of the full design, because it moves parts of that design outside the region of the design itself [2]. We suspect that the response of many readers would be to extend the pattern of 1(b) to the wallpaper design of 1(d), so the glide reflection would apply to the new design. But this wallpaper pattern contains *too many* symmetries, such as the vertical glide reflection shown by the dashed line. That glide reflection is not a "natural" symmetry of the original design. While the "partial" horizontal glide reflection in figure 1 was likely a deliberate decision on the part of the artist, it seems very *unlikely* that the vertical glide reflection in figure 2 represents the design intent of that artist.

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[1] These three authors did their work while students at Beloit College under the direction of D. Chavey. They also classified the two-color double strips. They are now at Assumption High School (IA), Google, and Rockton School District (IL), respectively.
[2] In group theoretic terminology, all motifs in figure 1c lie in one orbit, under the action of the symmetry group, while the motifs in 1b fall into two distinct orbits. See Chavey (2011) for details on this approach.

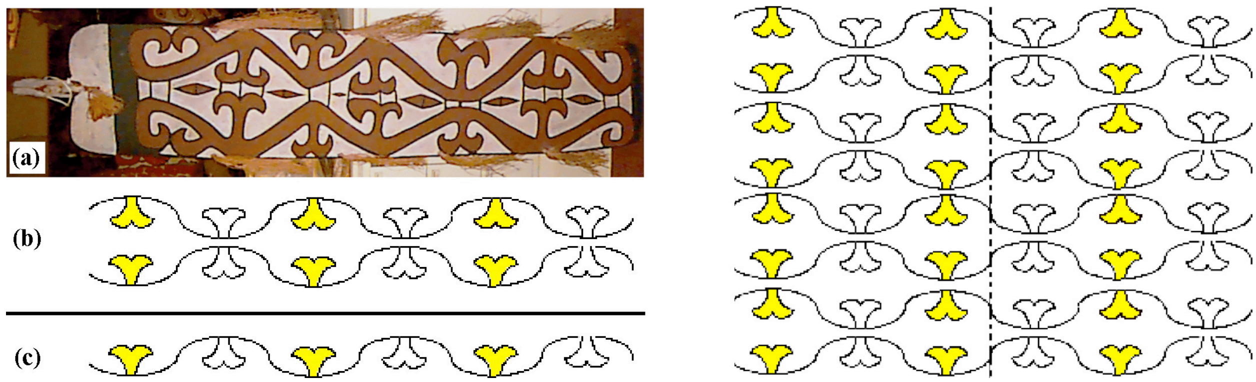

**Figure 1:** On the left is an example of a double strip design from a Papua New Guinea war shield, an abstraction of that design, and a single proto-strip from which the design could be constructed. On the right is an extension of the double strip pattern to a wallpaper pattern. Figure 1a is used with permission of the Logan Museum, Beloit College.

We thus define a "Double Strip Pattern" as two adjacent strips where each strip can be obtained from the other through an isometry (a distance-preserving movement) but where there are "*partial symmetries*" within the design, that is isometries that locally act like a symmetry, taking portions of the design to other equivalent portions, but which are not actually symmetries of the complete design. These partial symmetries fall into two types: (1) internal symmetries of a single strip that are not also symmetries of the full design; or (2) an isometry that moves one of the strips to the other but does *not* interchange the two strips, so that the second strip is moved out of the design. In case (1) the isometry becomes a symmetry if we limit the design to a *smaller* region than intended by the artist; and in case (2) the isometry becomes a symmetry if we extend the design to a *larger* region than intended by the artist. Figure 2 shows a double strip design containing both types of "partial symmetries". We discuss the various options for these two types of isometries in §3. In analyzing these designs, we refer to a single strip, i.e. half of the full design, as a "proto-strip" for the design.

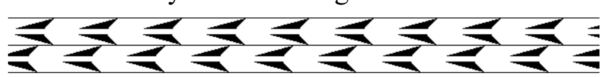

**Figure 2:** A double strip design with both types of "partial symmetries": A translation between the strips; and a horizontal reflection inside the proto-strips.

This definition does not include all designs that can be created by sewing together two copies of a pattern. In many cases, the true symmetries of such a design will include all of the symmetries of the proto-strips and all of the isometries that interchange the two proto-strips, so that there are no partial symmetries. In this case, we gain no additional information about the design that we did not get from its symmetry group. It is when there *are* such partial symmetries that this type of analysis tells us that there is more structured patterning within the design than is revealed by the symmetries alone.

In section 2, we discuss further our claim that extending a double strip pattern to a wallpaper pattern is an inappropriate approach to modeling these designs. We will then classify the types of double strip patterns that exist, give examples from ethnographic art of many such patterns, and look at related work.

## 2. Double Strip Patterns are *Not* Wallpaper Patterns

An obvious objection to the consideration of double strip patterns as truly *different* types of symmetric patterns is that we can generally extend such a design to a wallpaper pattern–patterns which are well-studied and understood. In fact, in [5] Washburn and Crowe explicitly state that when there are two rows of motifs in a design, we should view that design as part of a larger wallpaper design. However that book is especially focused on archaeological work, where we may have only a portion of the larger design from which to infer the pattern. In this case the extension to a wallpaper design may be well justified. The designs we study here, though, are ones in which we know that the artist has limited themselves to exactly these two strips, and gives us no reason to believe that an extension to a wallpaper design reflects their artistic intent. In ethnomathematics, it is natural to *try* to model this artistic intent, and hence we would not wish to embed such a design in a larger, or more symmetric, design than what was intended. While this is an artistic/cultural argument, there are several arguments from a purely mathematical viewpoint as to why we should avoid such wallpaper extensions. Problems includes the introduction of "unintended" symmetries in the completed design, mentioned above; concerns about ambiguity of the larger wallpaper pattern, so that two people would classify the same design in two different ways; and examples in which mathematically different double strip patterns would be classified as the same wallpaper design.

**Ambiguity:** If we choose to extend a double strip pattern to a wallpaper pattern, we may not get a unique extension. When there is a single partial symmetry, a double strip pattern can usually be extended to a wallpaper pattern that maintains the symmetries of the double strip, and for which that partial symmetry becomes a symmetry of the larger pattern. However, if there are multiple partial symmetries, it may not be possible to extend the design so that each partial symmetry becomes a symmetry. In this case, we must choose which partial symmetry to extend. With the design of figure 2, for example, extending it to make the horizontal reflection of the proto-strips into symmetries of a wallpaper pattern forces the pattern on the left of figure 3. The other partial symmetry is a translation that maps strip 1 to strip 2, and extending the pattern so *this* becomes a symmetry forces the pattern on the right of figure 3. There is no extension of figure 2 that turns both partial symmetries into symmetries. There is thus no natural "classification" of figure 2 by wallpaper patterns, since the choice of which category to put it in would be ambiguous.

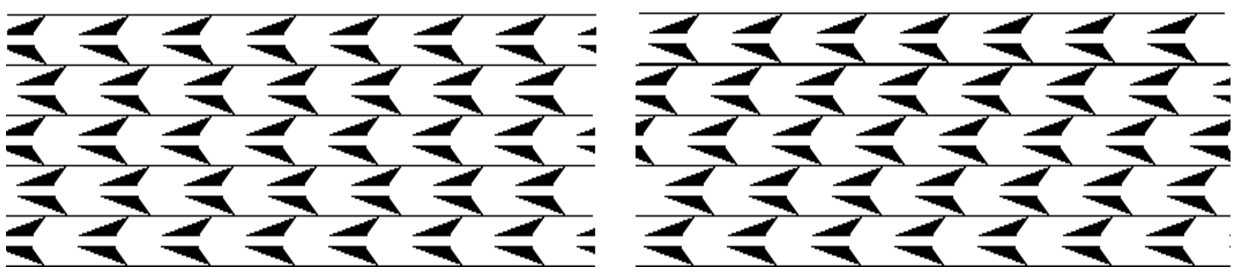

**Figure 3:** Two ways to extend one of the partial symmetries of the double strip pattern of figure 2 to a wallpaper design. These extended designs have *pm* and *p1* symmetry respectively.

**Consolidation:** It is possible to take two quite different double strip patterns, extend them to wallpaper patterns, and end up with identical designs. On the left of figure 4 are two double strip patterns that have different symmetry groups for the proto-strips *and* have different symmetry groups for the full design. Nevertheless, they both extend to the design on the right. Thus a "classification" of these designs by their containing wallpaper patterns would be unable to distinguish between the two patterns on the left. But those two patterns are mathematically quite different, hence such a classification would necessarily ignore some of the mathematical differences in those patterns while consolidating them into the same "class".

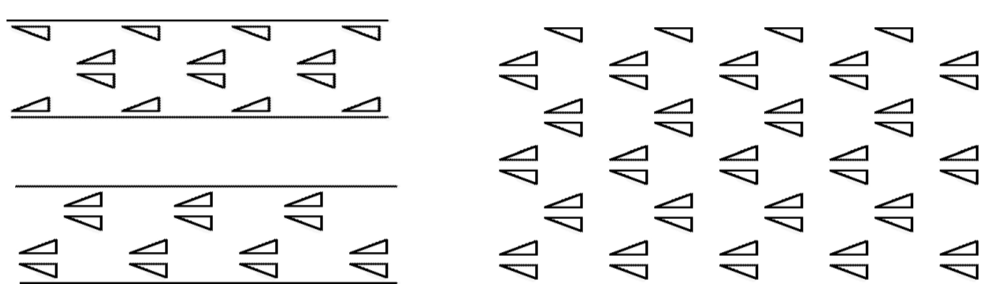

**Figure 4:** Two mathematically different double strip patterns (on the left) that each extend, in the natural way, to the *cm* wallpaper design on the right. The double strip patterns are shown to the left of a portion of the *cm* design that contains that double strip pattern.

## 3. Classification of Double Strip Patterns

To classify the types of double strip patterns, we focus on an isometry that moves the lower horizontal proto-strip to the upper one. (Whenever there is one such isometry, there will be infinitely many. We select one of them.) If this isometry is a symmetry of the combined double strip we call these "symmetric double strip patterns." This can happen via: (i) a (horizontal) reflection along the top edge of the proto-strip; (ii) a rotation of ½ at a point on the top edge of the proto-strip; or (iii) a horizontal glide-reflection along the top edge of the proto-strip. For such a design to be a double strip pattern, they must then have a symmetry of a proto-strip which is *not* a symmetry of the double strip. Alternatively, the isometry may be (iv) a translation with a vertical component, or (v) a vertical glide reflection. Either of these actions would move the lower proto-strip to the upper one, but would move the upper proto-strip off of the design, and hence will *not* be a symmetry of the double strip pattern. We refer to such designs as "non-symmetric double strip patterns". In several cases, the overall design can be constructed in two or more ways, hence some designs we find will already occur earlier in our lists. If there are *any* symmetric ways to construct a given design, we will include it with the symmetric double strip patterns.

In this section, we will classify, and label, the patterns according to these 2 main categories, and these 5 sub-categories. For each pattern type we discover, we shall label the design according to the symmetry group of the proto-strip, followed by the symmetry group of the double strip, using the format "double strip symmetry :: proto-strip symmetry".

### 3.1 Symmetric Double Strip Patterns

As mentioned, symmetric double strip designs must have a proto-strip symmetry that is *not* a symmetry of the full design. This will not happen if the proto-strip has type *p111*, since the translation is always a symmetry of both proto-strips. In case (i), with a horizontal reflection generating the top proto-strip, a proto-strip of type *pm11* will generate a design where all proto-strip symmetries are symmetries of the full design. In this case there would be no partial symmetries, hence we would not have a proper double strip pattern. The other 5 symmetry groups generate the double strip patterns labeled H1–H5 in figure 7.

If we rotate one proto-strip to create the second, the patterns we get from *pm11*, *p112*, *pma2*, or *pmm2* will depend on the location of the rotation point relative to the symmetry elements in the proto-strip. Figure 5 shows these four strip designs with their vertical reflections and rotation centers aligned on the two dashed lines. We may get different designs if the rotation point is: (a) on one of those two lines; (b) exactly half way between those lines; or (c) at any point other than those two. If the proto-strip has symmetry *pm11* and the rotation point of the combined pair of strips is aligned with either vertical reflection of the proto-strip, then the vertical reflection is a symmetry of the full design, and again does not generate a double strip pattern (see figure 6). For the symmetry groups *p1a1* and *p1m1*, the three cases (a)–(c) described above give mathematically equivalent double strip patterns, R2 & R3 below. With *pma2* and *pmm2*, case (a) gives the same designs as H4 and H5, hence are not listed separately in figure 7. The other two cases each give distinct designs for each proto-strip: R4–R7 below. (With both of these groups,

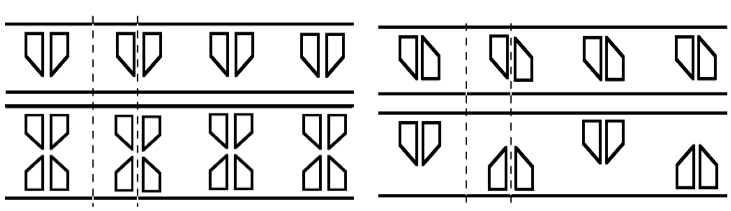

**Figure 5:** *pm11*, *p112*, *pmm2*, and *pma2* strip designs with the locations of vertical reflections and rotations aligned on the dashed lines.

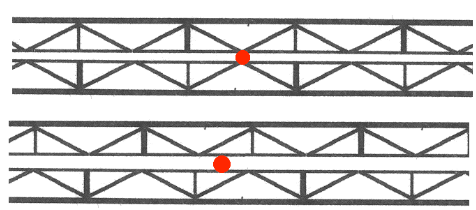

**Figure 6:** A *pm11* proto-strip rotated about either marked rotation point gives a regular strip pattern, with no partial symmetries.

case (a) gives additional symmetries to the double strip other than just the built-in rotation.) Finally, with *p112*, the three cases all give designs with symmetry *p112* :: *p112*, hence from a symmetry group viewpoint they are equivalent designs. However, if one looks at the relative placement of the various rotation points (in each proto-strip, and in the double strip), these designs are geometrically distinct. In this situation, all three cases give strips that naturally extend to *p2* wallpaper groups (see chart 1), but in other situations the distinction between the three cases result in strips that expand to different symmetry types of wallpaper designs. Consequently, although we list them all as type R8, we also distinguish them in figure 7 as R8(a)–R8(c).

Because we want the isometry which takes one proto-strip to the other to be extendable to a wallpaper pattern, a horizontal glide reflection which generates a double strip pattern from a proto-strip must glide half the length of the translation, i.e. must be a glide reflection of the double strip pattern itself. Once again, if the symmetry group of the proto-strip is either *p111* or *pm11*, the resulting combined pairs of strips will be a strip pattern with no partial symmetries. For *pma2* and *pmm2*, this glide reflection is identical to what we get from a rotation, i.e. generating patterns R4 and R6 already on the list. The other three strip symmetry groups give us new double strip patterns, shown in figure 7 as HG1–HG3.

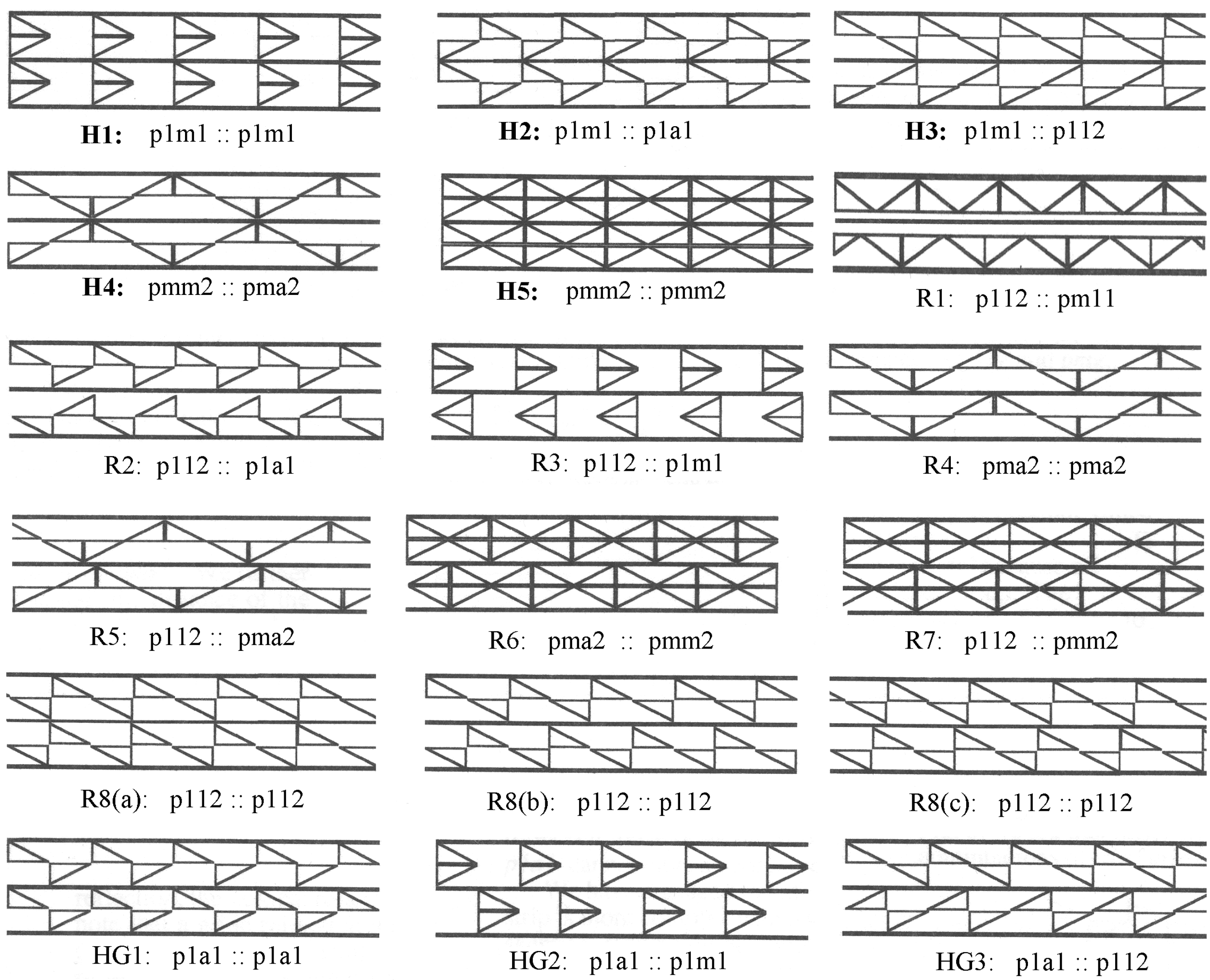


**Figure 7.** The 16 symmetric double strip patterns generated from a proto-strip by an internal symmetry. These designs were classified by the last 3 authors as an undergraduate project directed by the first author. They also classified all 2-color symmetric double strip patterns, not shown here.

### 3.2 Non-Symmetric Double Strip Patterns

A proto-strip can generate a double strip via a translation from the lower proto-strip to the upper one. If the proto-strip has rotational symmetry, then the combination of that rotation with the translation will be a rotation at the edge of the proto-strip, and hence were classified in the previous section. We thus get no new double strip patterns from proto-strips of types *p112*, *pma2*, or *pmm2*. With the other strip symmetry groups, we need to distinguish the three cases where the horizontal component of the translation is (a) zero; (b) ½ the translation length; or (c) some other value $x$, $0 < x < ½$. If the proto-strip contains a horizontal reflection or glide-reflection symmetry, then the combination of that with a translation of case (a) or (b) will be a horizontal reflection or glide reflection that moves one proto-strip to the other. These cases have also been previously classified. Thus when the proto-strip is *p1m1* or *p1a1*, we will get new double strips patterns from case (c) only, shown as T5–T6 in figure 8. With *p111* and *pm11* we will get new designs in all three cases. With *p111*, the three designs are equivalent from a group theory viewpoint, but quite different from a geometric viewpoint, so we list them as T1(a)–T1(c). With *pm11* the three cases are quite different, e.g. they extend to three different wallpaper groups: *pm*, *cm*, and *p1* (respectively).

The final way to generate a double strip pattern is via a vertical glide reflection, which moves the lower proto-strip up, and then reflects it in place. If the proto-strip has either a horizontal or glide-reflection symmetry, then that symmetry would combine with the vertical glide reflection to form a rotation at the edge of the proto-strip, generating a symmetric double strip pattern classified above. If the proto-strip had a vertical reflection, then it would combine with the vertical glide reflection to form a translation from one proto-strip to the other, generating a design classified in the previous paragraph. This eliminates all but *p111* and *p112* as the possible symmetry types of the proto-strip. With *p112*, the combination of the vertical glide-reflection and the rotation will form a horizontal glide-reflection along the proto-strip edge. This case was classified *if* the length of that horizontal glide reflection was half the length of the translation, which will happen if the vertical glide reflection line goes through the rotation point or is half-way between them. When the vertical glide reflection line is located elsewhere, *p112* gives us a new design, shown as VG1 in figure 8. Finally, *p111* will generate a new type of double strip pattern via a vertical glide reflection. *p111* has no fixed locations of symmetry elements, so there is no true equivalent to the three cases of the translations, and it generates the design VG2 of figure 8.

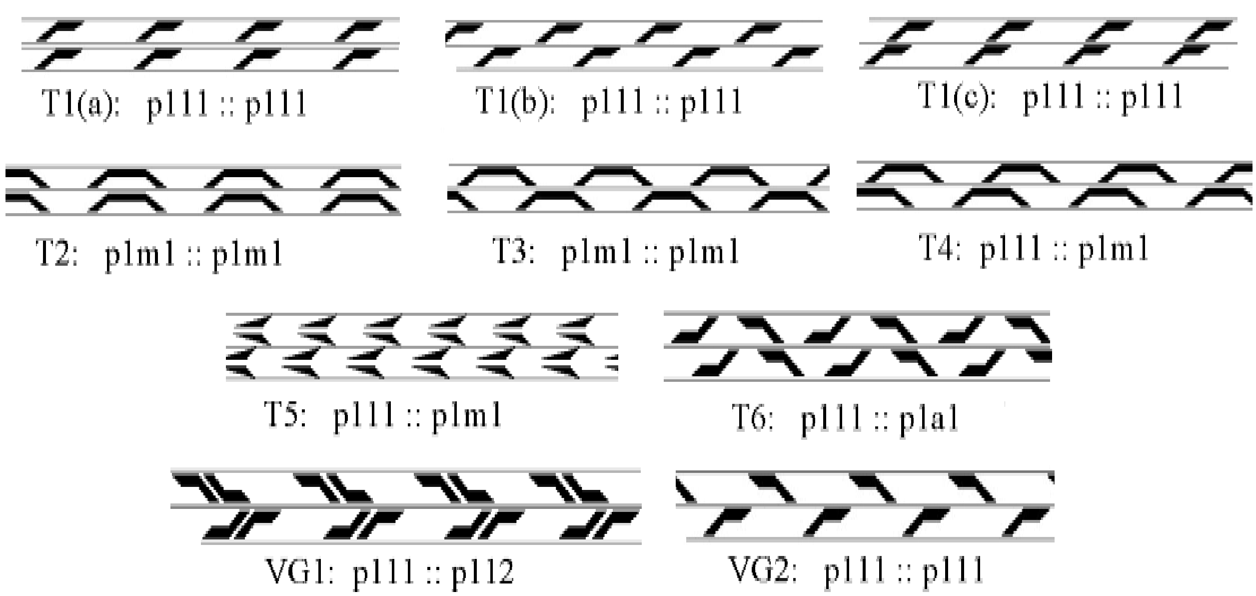


**Figure 8.** The 7 non-symmetric double strip patterns. (The first three are mathematically equivalent, but artistically distinct.) These designs have no symmetry that interchanges the top and bottom proto-strips.

## 4. Example: Andean Art

Earlier, we gave an example of a double strip pattern of type H2, from Papua New Guinea. The Logan museum at Beloit College contains many examples of Papua New Guinea art with double strip patterns, almost all of which are of this type H2. (See [3] for several examples.) In contrast, our investigation of Andean art reveals a surprisingly large diversity of double strip patterns. We looked at several books on the art and archaeology of the Andean cultures, and found 45 examples of double strip patterns from 500 B.C.E. through the Spanish conquest, and across several Andean cultures. Table 1 shows the distribution of designs found, and lists the ways that each double strip pattern can be extended to a wallpaper pattern by extending one of its symmetries or partial symmetries. This list of Andean patterns includes 16 of the 27 possible designs (since they are not mathematically distinct, we do not separate the three types of T1 patterns in this chart). This would appear to imply that this is a reasonable categorization of such designs: It applies to a substantial number and variety of designs, but does not include too many unused classes.

| Double Strip Design | Embeds in Wallpaper Designs | Samples from Peru |
|---|---|---|
| R1 | *p2 or pgg* | 0 |
| R2 | *pgg* | 1 |
| R3 | *pmg* | 0 |
| R4 | *pmg* | 5 |
| R5 | *p2* | 0 |
| R6 | *cmm* | 0 |
| R7 | *p1* | 0 |
| R8(a) | *p2* | 2 |
| R8(b) | *p2* | 2 |
| R8(c) | *p1* | 0 |

| Double Strip Design | Embeds in Wallpaper Designs | Samples from Peru |
|---|---|---|
| H1 | *pm* | 1 |
| H2 | *cm* | 0 |
| H3 | *pmg* | 1 |
| H4 | *pmm* | 6 |
| H5 | *pmm* | 6 |
| | | |
| HG1 | *pg* | 0 |
| HG2 | *cm* | 2 |
| HG3 | *pgg* | 0 |

| Double Strip Design | Embeds in Wallpaper Designs | Samples from Peru |
|---|---|---|
| T1 | *p1* | 8 |
| T2 | *pm* | 1 |
| T3 | *cm* | 2 |
| T4 | *p1* | 3 |
| T5 | *p2 or pgg* | 0 |
| T6 | *p1 or pm* | 2 |
| T7 | *p1 or pgg* | 0 |
| | | |
| VG1 | *pg* | 1 |
| VG2 | *pg* | 2 |

**Table 1:** Number of designs of each type found in Andean art.

One design noticeable by its absence in this chart is H2–the design type predominant in the similar art of Papua New Guinea. This implies that there are cultural differences at work in affecting which types of patterns are selected. We see two possible Andean sources of this: Moche fashion, and the architecture of the Inca and their predecessors. The Moche built a large number of "portrait vessels", pottery that incorporated images of various people. A Google image search for "Moche portrait vessel" will bring up hundreds of such vessels, the vast majority of which wear head scarves wrapped like that in figure 9, a solid majority of them being wrapped twice around the head to form a double strip pattern. In addition to being a source of many such patterns, this fashion element may have influenced the art of later cultures.

Another possible source for artistic inspiration for these designs may have come from the extensive architectural constructions of both the Inca and other Andean cultures from whom they inherited some of their techniques (see figure 10). When you are building brick walls, ones that must resist occasional

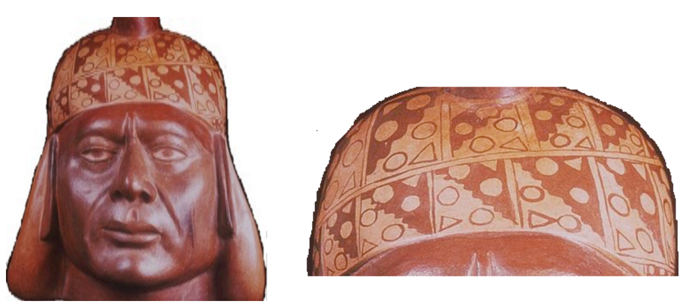

**Figure 9.** Moche portrait pottery vessel, with a close-up of the head scarf being worn. This design is an example of the VG1 double strip pattern. This picture is used with permission of UC-San Diego.

earthquakes, you want to stagger bricks in adjacent rows. Design types H1 and H2, if used in constructing walls, would build particularly weak walls. Architectural construction would also encourage building row upon row of equivalent structures, which may explain the frequency of the translation types T1–T7 in their art. More research would be necessary to decide if either of these guesses were a viable explanation for their design preferences. But the vast differences between the design types of Papua New Guinea and those of the Andean cultures argues for a strong cultural component to the types of designs created.

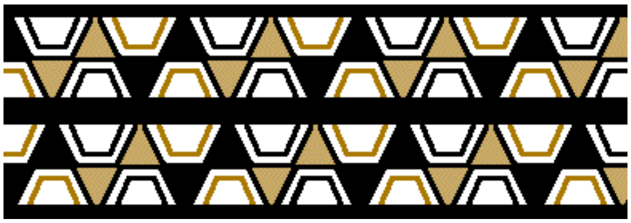

**Figure 10.** An example of a double strip pattern of the Inca, demonstrating the influence of Incan architecture, including the use of the Inca trapezoid, on the artwork of that culture. (Redrawn from Ascher, [1].)

## 5. Previous Work on Double Strip Patterns

In [2], Bérczi analyzes double strip patterns in Hungarian designs. He starts with a single strip pattern, looks at the individual repetition units, and imagines doing a transformation of each repetition unit into the region where the second strip should lie. This does not generate all the patterns of this classification, esp. those that involve a horizontal offset of the first strip to construct the second. Those types of designs are important in Andean art, so his classification is not appropriate for this context. His tactic does describe some "double frieze patterns" that our classification does not include, but neither he nor we have examples of any of those designs from historic artwork.

In [4] Grünbaum analyzes what he calls "ribbon patterns," which include nearly 50 types of designs that we would view as double strip patterns. He is classifying combinatorial regularity, as opposed to symmetry regularity, which is the reason for the additional types. As with Berczi, the additional types of designs he finds did not arise in our artifact search, and Grünbaum does not give data about how many of his designs correspond to real-life artifacts. Some of our designs where the proto-strips are offset, but by less than ½, are not included in his catalog, and these designs *did* arise in our survey. An argument that Grünbaum makes is that too often mathematicians obsess over expecting symmetry groups to model actual designs, when in many cases the designs are too complex for such a restrictive structure. While we differ in our approaches to expanding design classifications beyond the fundamental symmetries of the design, our papers share the goal of looking for ways to understand pattern regularity that is more complex than can be modeled by their symmetry groups.

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