Infinite Rhythmic Tiling Canons

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Abstract
Maximally-even (ME) rhythms, in which attacks (or the onsets of notes) are distributed as evenly as possible over a given number of pulses, are common in many cultures. One way to generate a ME rhythm with $m$ attacks per $n$ pulses is to digitize a line of slope $m/n$. If the slope is rational, the resulting rhythm is periodic; irrational slopes yield Sturmian sequences, which are balanced, almost periodic, self-similar and hierarchical sequences that are highly relevant in the study of musical objects possessing these same properties. Rhythmic tilings are combinations of rhythms that yield one and only one onset per pulse. Rhythmic tiling canons are tilings in which the component rhythms are version of a single tile. Sturmian rhythms can be used to create completely new kinds of aperiodic rhythmic tiling canons, in which the relations of the component rhythms are determined by the continued fraction of the slope.

Sturmian Rhythms
Maximally-even (ME) rhythms, in which attacks are distributed as evenly as possible over a given number of pulses, are common in many cultures. One way to generate a rhythm with a ME distribution of $m$ attacks over $n$ pulses is to take the lower mechanical sequence of slope $a = 1 - m/n$, a discretization of the line $y = ax$. Given $a \in \mathbb{R}$, the lower mechanical sequence of slope $a$, $c_a : \mathbb{N} \rightarrow \{0, 1\}$, is given by

$$c_a(n) = \left\lfloor (n + 1)a \right\rfloor - \left\lfloor na \right\rfloor - \left\lfloor a \right\rfloor.$$

If the slope is rational, this sequence will yield a periodic ME rhythm. For instance, if $a = 5/9$, then the sequence will be a repetition of 0101101011011... Associating attacks with 0s in the sequence yields the 2 + 2 + 2 + 3 rhythm of Dave Brubeck’s Blue Rondo à la Turk, which is a ME distribution of four attacks over nine pulses.

Irrational slopes, where $a \in \mathbb{R} \setminus \mathbb{Q}$, yield Sturmian sequences, which are balanced, almost periodic, self-similar and hierarchical sequences that are highly relevant in the study of musical objects possessing these same properties. For example, the Sturmian sequence of slope $\phi$ is $c_\phi = 01011010110111...$, which can be separated into groups beginning with each 0. These groups are of length two or three (Figure 1).

These groups can be separated into hierarchical levels containing short and long groups as shown in Figure 2. Note the while the durations of level one are equal, at higher levels the ratio of long to short durations converges on $\phi$. The sequence of events at each level is an augmentation of the sequence of the previous level. This is possible because of the periodicity in the continued fraction for $\phi$.

Rhythmic Tiling Canons
A rhythmic tiling is a combination of rhythms in which one and only one onset occurs per pulse. A rhythmic tiling canon is a tiling in which the component rhythms are version of the same rhythm, transformed
by simple delay, scaling (augmentation or diminution), and/or retrograde \(^1\). In Figure 3 a simple rhythm is combined with a delayed version to create a common rhythmic tiling canon\(^2\).

Figure 3: The rhythmic pattern or tile in (a) combines with a delayed version of the tile to form a rhythmic tiling canon in (b).

Sturmian rhythms can be used to create completely new kinds of aperiodic rhythmic tilings, in which the canonic structure of the component rhythms is determined by the continued fraction of the slope \(2\). The typical rhythmic tiling is periodic with tiles of finite length. For tilings based on Sturmian rhythms both tiles and tiling are aperiodic and the composite rhythm is often a diminution of one or more tiles.

Figure 4 shows the rhythmic structure for a tiling canon by Tom Johnson in which successive augmentations are delayed so as to fill in gaps left by previous voices. This tiling pattern can be adapted to Sturmian canons as demonstrated in Figure 5. Successive voices are successive levels of the sequence \(c_2\) with onsets for the shorter durations and rests for the longer durations. As with Johnson’s canon, successive voices are delayed so as to fill gaps left by previous voices. Gaps in Figure 5 would be covered by higher-level voices.

Figure 6 shows an original infinite tiling canon based on the Sturmian sequence of slope \(\phi\). Successive voices are progressive near augmentations of previous voices. Some attacks are omitted in the canon line for musical purposes. Nonetheless, by the end of the excerpt the eighth-note pulse is fully tiled by the

\(^1\) This pattern is the drum rudiment known as the “paradiddle” and occurs throughout much of Steve Reich’s Different Trains.
The combination of the canon line proceeding at four different speeds.

**Figure 4**: Rhythmic structure for tiling canon from one of Tom Johnson’s Rational Melodies. Lower voices are progressive augmentations by a factor of three and are delayed so as to fill the gaps left by higher voices. The unison rests at the end of the excerpt would be the beginning of the fourth voice.

**Figure 5**: Tiling pattern from Figure 5 adapted for \(c \sqrt{r}\).

References


Figure 6: Original tiling canon based on a Sturmian sequence of slope $\phi$. By the time the fourth voice enters, the composite rhythm of all four voices yields a sequence of continuous eighth notes.