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# Yvon-Villarceau Circle Equivalents on Dupin Cyclides

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## Abstract

A torus contains four families of circles: parallels, meridians and two sets of Yvon-Villarceau circles. Craftworks and artworks based on Yvon-Villarceau circles can be very attractive. Dupin cyclides are images of tori under sphere inversion, so they contain the images of the torus circles families. I applied operations that are known to create effective artworks on tori to Dupin cyclides, and proved them to be feasible. The regularity and the hidden complexity of the objects I obtained make them very attractive. Reviving the 19th century's tradition of mathematical models making, I printed several models, which can help in understanding their geometry. The tools I developed can be generalized to explore transformations of other mathematical objects under sphere inversion. This exploration is just at its beginning, but has already produced interesting new objects.

## 1 Introduction

Circles and spheres are among the simpler mathematical objects. Their regularity, their compact definition make them suitable elementary bricks for a computer artist to manipulate, either from the point of view of the programmer or of the artist. I always try to find mathematical objects composed of circles or spheres which I can represent, manipulate, parameterize and modify. Mathematical transforms that preserve or create circles are also very useful tools when dealing with objects made from circles or spheres. They allow us to further modify objects that are already attractive. At the crossroads of these two constraints (dealing with circles, transforming circles), we find the family of Dupin cyclides: as images under inversion of basic mathematical objects (cylinders, cones, tori), they have all the good properties: they are nice objects *per se*, they can be defined by families of circles, they look fine when outlined as wireframes, they look very different when seen from various points of view, they are easy to print. The rest of the paper is organized as follows: the next two sections define inversion and Dupin cyclides. The following part introduces Yvon-Villarceau circles, a family of circles defined on torus, and shows how we can use them to create new objects. The last section describes a generalization of the process for other mathematical shapes.

## 2 Inversion

In the plane, circle inversion is a transform defined by a circle centered at point $O$ and with radius $r$. The image of a point $P$ is the point $P'$, colinear with $O$ and $P$ and satisfying $|OP| \times |OP'| = r^2$. Sphere inversion is the generalization of circle inversion in three dimensions space. After a sphere inversion, the part of the space that was outside the inversion sphere is mapped to the inside of the sphere, and vice-versa: any object that was lying far away is made close to the center, when objects initially close to the center of inversion are sent far away, and are distorted. By sphere inversion planes and spheres are mapped to planes and spheres, circles and lines are mapped to circles and lines. Inversion also preserves tangency. Those nice properties can be used to transform a simple object composed of circles and lines into a more interesting design that retains many properties of the original object. I have previously used these properties to produce circle packing compositions [3]. Figure 1 illustrates some properties of circle inversion. The circle of inversion is $C$ with center $I$. Since inversion is an involution, objects labeled with the same letter, upper or lower case, are reciprocal images under the inversion related to $C$. Couple $(A,a)$ shows an example where an external circle is mapped to an internal one. $(B,b)$ shows that a circle that intersects $C$ is mapped to another intersecting circle. Finally, $(D,d)$ illustrates the fact that a circle passing through the center of inversion is mapped to a line, and vice-versa.

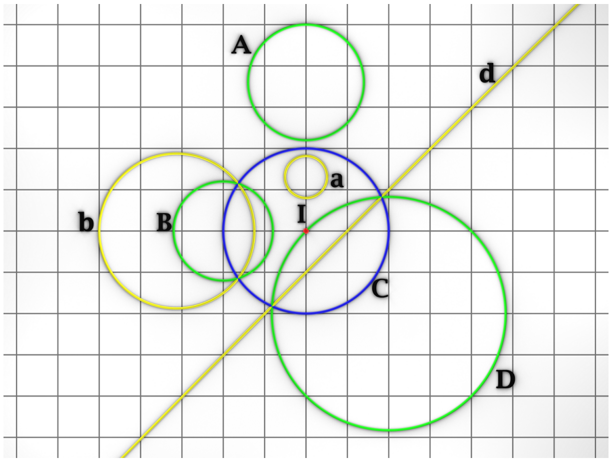


**Figure 1** : *Different effects of inversion, according to the respective positions of the circles.*

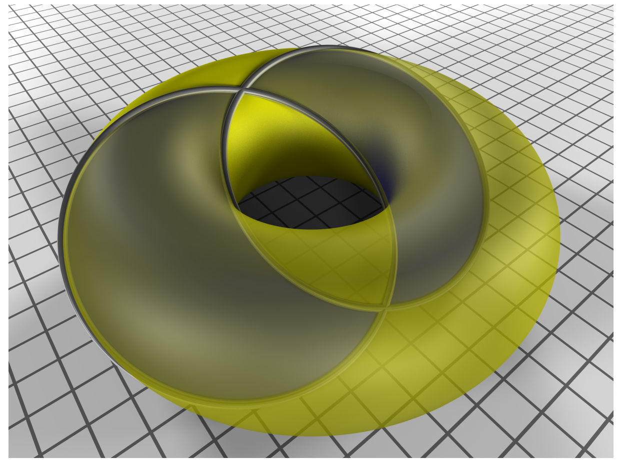

**Figure 2** : *Cutting a torus to create two Yvon-Villarceau circles.*

## 3 Dupin Cyclides

Inversion also serves as a way to define a family of curves: the Dupin cyclides. Dupin cyclides are named after Pierre Charles François Dupin, a 19th century French mathematician. Dupin cyclides are algebraic surfaces of degree three or four, with all lines of curvature being circular. There are several definitions of cyclides, Chandru et al.[2] list six of them. A seventh definition is based on inversion: Dupin cyclides are images of cylinders, cones or tori under inversion. According to the position of the center of the inversion sphere $O$, there are thirteen different cyclides [9], some of them infinite. Since they contain differing amounts of curvature, they have recently regained interest in Computer Aided Design, because they can be used as patches to smoothly connect curved surfaces [6], just like the *French curve* tool was used to connect lines in the 2D plane. Figures 3 to 10 illustrate different kinds of cyclides, depending of the surface being inverted, and of the position of the center of inversion relatively to this surface. Cylinders generate four kinds of cyclides, two of them being represented in figure 3 and figure 4, respectively when the center of inversion is inside the cylinder (cuspidal cyclide) and when it lies outside the cylinder (needle cyclide). Cones generate six kinds of cyclides, called horn and spindle cyclides (two examples in figures 5 and 6). Tori generate three new kinds of cyclides (ring cyclides): two of them are shown in figures 7 and 9. See [9] for more details. Note that in figures 3 to 6, there should be singular points, which are shown with a non-null thickness, because of the method used for rendering these objects.

Cyclides are very interesting for at least two reasons.

First, from an artistic point of view, Dupin cyclides have weaker symmetry than spheres or tori, but on the other hand a kind of *regular irregularity* which attracts the eye.

Second, from the point of view of the programmer, coding cyclides is a small challenge, and writing this code properly is a good exercise.

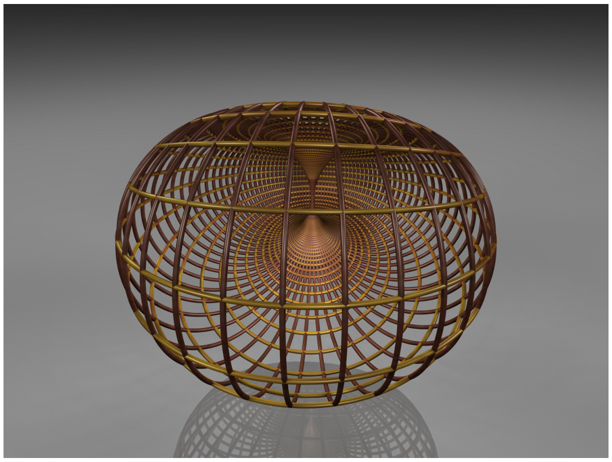

**Figure 3** : *Inversion of a cylinder: cuspidal cyclide.*

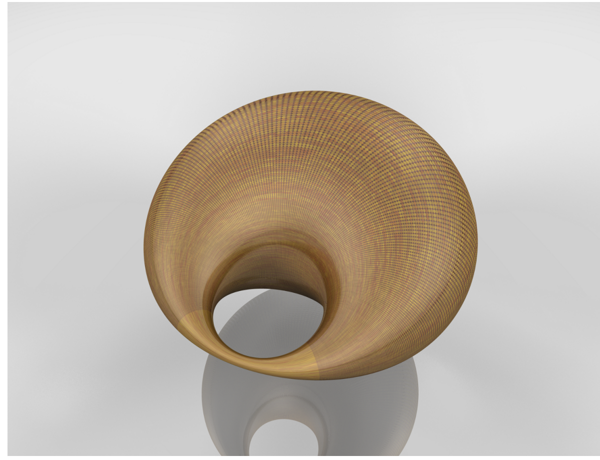

**Figure 4** : *Inversion of a cylinder: needle cyclide.*

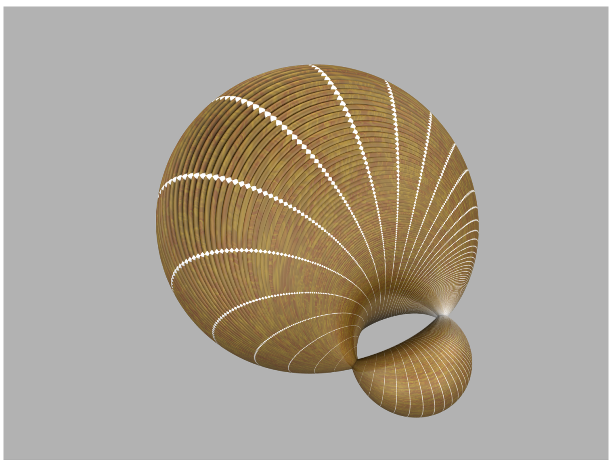

**Figure 5** : *Inversion of a cone: horn cyclide.*

**Figure 6** : *Inversion of a cone: spindle cyclide.*

## 4 Yvon-Villarceau Circles and their Images

Under sphere inversion, all the circles and lines included on a surface to be inverted will be mapped to circles or lines, depending on whether they contain the center of inversion or not. Cylinders and cones have one family of circles and one family of lines, which generate two families of circles on the inverted surface (figures 3 to 6). Tori have two perpendicular families of circles corresponding to the circles and lines on cylinders and cones, but they have also two other sets of circles: the Yvon-Villarceau circles, obtained by cutting the torus with a doubly tangent plane (figure 2). These circles are in turn transformed (using inversion) in circles on the surface of the cyclide. This fact opens a wide range of possible artworks and hands-on mathematical activities. The equations of the images of Yvon-Villarceau circles on a cyclide have been computed by Lionel Garnier [6]. This helped me to create images (figure 7) and 3D-printed versions of ring cyclides (figure 8). On these models, all four families of circles are represented. When the center of inversion lies on the torus, the resulting cyclide is infinite, but we can still create representations of finite parts of it, for example by bounding the cyclide by a sphere. This is the method I used to create the virtual (figure 9) and 3D-printed (figure 10) versions of the parabolic ring cyclide. Figure 10 presents an interesting feature, namely the four colored straight lines: because the center of inversion is on the surface of the torus, it belongs to four circles. Since a circle passing through the center of inversion is mapped to a line, each resulting parabolic ring cyclide contains exactly four lines, which are highlighted on this model.

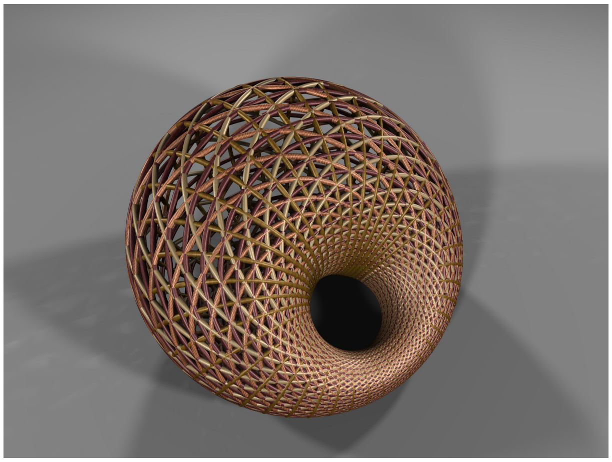

**Figure 7** : *A virtual ring cyclide with four families of circles.*

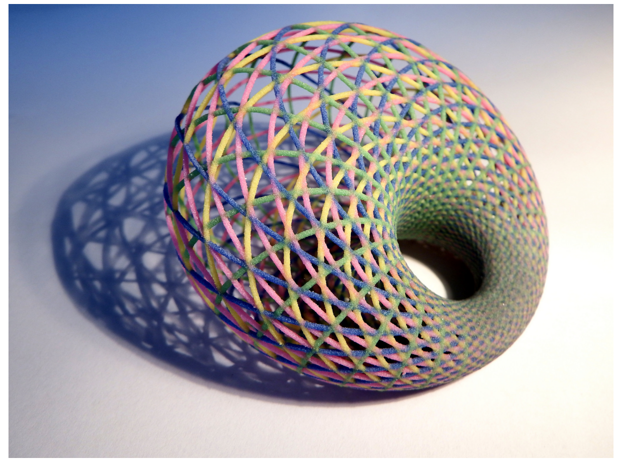

**Figure 8** : *A 3D printed ring cyclide with four families of circles.*

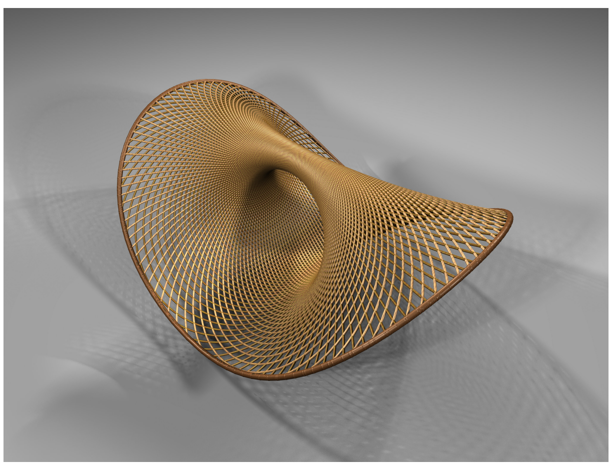

**Figure 9** : *Part of a parabolic cyclide outlined with its Yvon-Villarceau circles (virtual).*

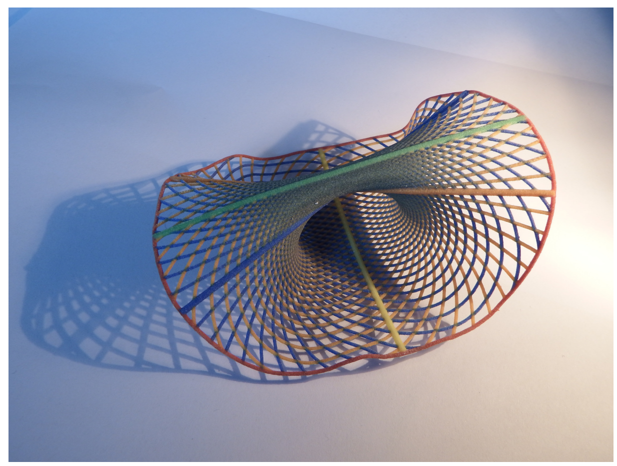

**Figure 10** : *Part of a parabolic cyclide outlined with its Yvon-Villarceau circles (3D printed).*

## 5 Slide-together

A slide-together construction is an object obtained by cutting paper pieces and slide them together, without glue [7]. On a torus, Yvon-Villarceau circles are at the basis of some interesting slide-together patterns. Maria Garcia Monera and Juan Monteverde [5] described an effective method to design a torus composed of interlocked Yvon-Villarceau-based moon-shaped parts. This method was discovered independently at the same time by Prof. Yoshinobu Miyamoto, who made a pattern available [1]. Figure 11 shows the M.G. Morena model. In order to see if such a construction is possible using Yvon-Villarceau circle images on a cyclide instead of *croissants*, I designed a model using some amount of trial and error rather than direct calculation in order to show feasibility. Further formalization should in the future automatize the process. This shows, at least, that it is worth working in that way. Patterns and assembly instructions are available on my webpage.

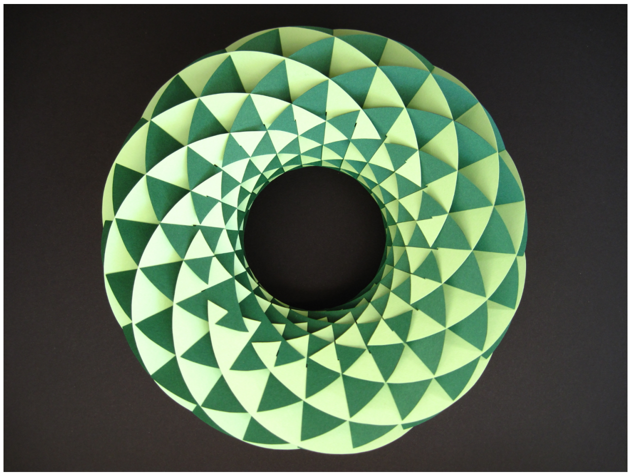

**Figure 11**: *A slide-together on a torus, using Villarceau moons (photo M. Garcia Morena).*

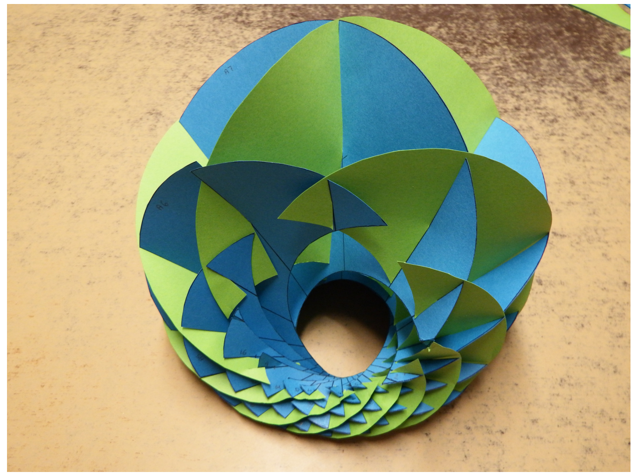

**Figure 12**: *A slide-together using non-uniform moons on a ring cyclide.*

## 6 Using the Tools for Other Surfaces

In the process of trying to physically illustrate cyclides and Yvon-Villarceau circles, I developed several tools such as programs in Java, scripts in Blender, to implement inversion and representations or manipulations of circles in three dimensions. It seemed interesting and possible to use those tools in a more general way than just cyclides. Inversion can be applied to every object defined with circles and lines: ruled surfaces, naturally, but also circle packing constructions, cardioidal constructions [4] etc.. Figures 13 and 14 demonstrate the potential of this approach. Since these objects only use circles in their description, they are easy to define and to describe in order to 3D-print them.

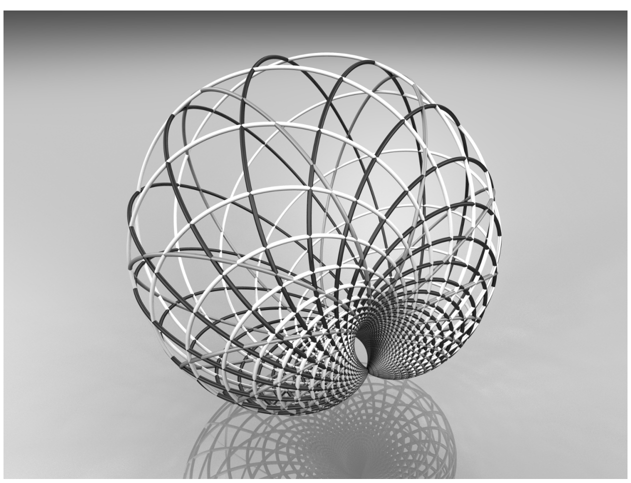

**Figure 13**: *Inversion of a hyperboloïd.*

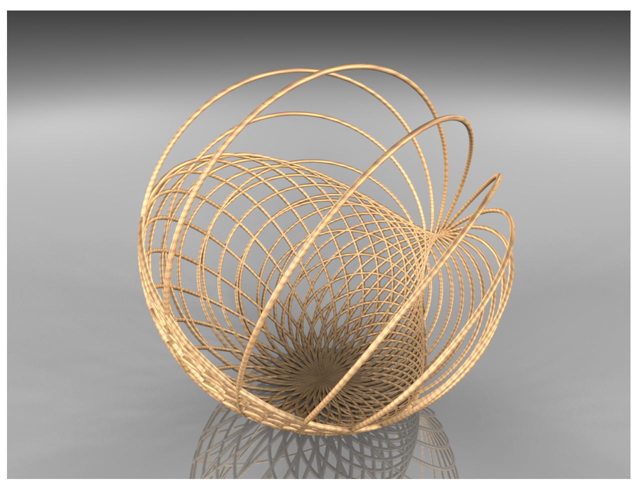

**Figure 14**: *Inversion of a string art cardioïd.*

## 7 Conclusion and Future Works

Exploring the family of Dupin cyclides proved to be fruitful and promising. I demonstrate one can make attractive images and fascinating 3D-printed objects. The creation of real-word objects (similar to Gabriela Ligenza's 3D-printed hats based upon my cardioidal objects [8][4]) is still to come, but it might just be a question of time. The methods and tools used in this work can and will be applied to other objects, such as ruled surfaces and Darboux cyclides, to explore new kinds of objects defined with circles. A lot of work is still to be done while exploring the results of various parameter settings: for each original surface or object, there are a lot of interesting positions where the center of inversion can be placed. Each of these positions can lead to very different final shapes. And each shape in turn presents different angles of view. A huge number of pictures, videos or 3D printed sculptures are still to be computed.

## References

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