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"In an Ocean of Ashes": Order and Chaos in Mathematics and Literature

Katharine Ahrens
Student, Dept. of Mathematics, Ithaca College
953 Danby Road, Ithaca, NY 14850, USA
kahrens1@ithaca.edu

Abstract

This paper explores the connections between math, as seen in the interplay of chaos and fractals, and literature, as seen in the form and structure of Dante's *Divine Comedy* and Stoppard's play *Arcadia*. Dante seems to intuitively grasp the order in the chaos that Benoit Mandelbrot and those that followed after him would formalize within fractal theory and dynamical systems theory centuries later. Stoppard is a separate case, showing that order and chaos are human themes, and also by bringing mathematics directly into his play. Stoppard's intentional and un-intentional connections serve as a bridge between the mathematics of fractals and the literature of Dante, and show that these two disparate disciplines connect on a fundamental and human level.

Introduction

The question of the relationship between order and chaos is a fascinating one. At first glance, these are two entirely opposite concepts. Indeed, this is how they are conventionally seen, as the Merriam Webster Dictionary defines chaos as "complete confusion and disorder" [9]. And yet, order and chaos are often interrelated and intertwined. The pioneering work of Benoit Mandelbrot in fractal and dynamical systems theory shows this; an orderly picture is created from a probabilistic algorithm or a simple, orderly equation results in mathematical chaos. This is relatively new mathematics.

And yet, artists have been attempting to make sense of this connection for centuries. Two of the works that show this order and chaos relationship will be discussed in this paper. Firstly, we consider Dante's *Divine Comedy*, specifically part one of the three-part poem, the *Inferno*, in which Dante tackles the enormous question of what happens after death. Dante's view of the afterlife is as something both neatly patterned and orderly and as something utterly disorderly. Secondly, we consider Tom Stoppard's *Arcadia*, a masterful play that addresses the relationship between past and present which also speaks of the problems of determinism and chaos directly. Both of these works reveal that despite the overwhelming chaos of life events, there is some beautiful and significant order to be found within them; Dante and Stoppard reveal this not only through the events they write about, but by the very structure of their works themselves.

While Dante of course could not have possibly been aware of Mandelbrot's work, Dante seems to intuitively grasp the order in the chaos that Benoit Mandelbrot and those that followed after him would formalize within fractal theory and dynamical systems theory centuries later.

Stoppard is a different case, because he wrote his masterful play *Arcadia* two decades after Mandelbrot's ground-breaking work. He reveals the intuition (like Dante gives us) and the rigor (that Mandelbrot and the mathematics give us) all within one play. In this way, Stoppard is really the "bridge" between mathematics and literature. We will also show some of the mathematics which Stoppard only alludes to in the play.

Dante, the Accidental Mathematician

First, we shall look at the various ways Dante exhibits mathematical thought in the Comedy. The first of these comes with the idea of patterns. Ninteenthth century mathematician G.H.Hardy [3] drew a connection between math and poetry in general, saying "a mathematician, like a painter or a poet, is a maker of patterns." In the case of Dante, Hardy's rule is certainly true. Dante divides the Divine Comedy into 3 parts, each with 33 cantos, except *Inferno*, which has 34. The slight irregularity makes the number of cantos sum to 100, which literary critic Prue Shaw [6] notes the medieval readers considered the perfect number because $1 + 2 + 3 + 4 = 10$, and $100 = 10^2$. Shaw mentions that Dante's organizational scheme makes the poem "not just a verbal artifact but a mathematical one."

The presence of patterns continues across cantos as well. Canto 5 of *Purgatorio* and *Paradiso* both contain echoes of Francesca da Rimini's story in Canto 5 of *Inferno*; the theme of a teacher appears in Canto 15 of all 3 parts; and some kind of a reference to Ulysses shows up in every Canto 26. However, if Dante is fascinated with patterns, we must then wonder why he doesn't make every canto in *Paradiso* and *Purgatorio* echo its counterpart in *Inferno*. Wouldn't that make the patterns more apparent? The answer, again, lies in how mathematical theorems operate. G.H. Hardy [3] said that "a property common to too many objects can hardly be very exciting, and mathematical ideas also become dim unless they have plenty of individuality." Repeating ideas through every Canto 1, Canto 2, Canto 3...would surely be unexciting. But like a mathematician, Dante recognizes that patterns are necessary, but they are boring if they are overly repetitious. Having only some cantos echo previous cantos makes sure this delicate balance remains intact.

Dante is also fascinated with geometry. Circles appear throughout the entire work, as there are circles of Hell, circular terraces of Purgatory, and spheres of heaven. In Canto 33 of *Paradiso*, Dante uses circles to express two very complicated ideas. The first is Dante's representation of the Trinity [2], which he sees as "three circles/having three colors but the same extent" (III.33.116-117). Dante is trying to describe the indescribable, and his best attempt to describe it is in the language of geometry. Dante also describes his confusion and inability to comprehend God in terms of geometry, saying he is "like the geometer who fully applies himself/ to square the circle and, for all his thought/cannot discover the principle he lacks" (III.33.133-135). The "squaring the circle" problem was a famous problem in medieval geometry which was not proven to be impossible until 1882. The description of the struggle to prove or explain something that seems possible but isn't working out is a problem every mathematician faces, and a struggle Dante feels too. He sees his problem of describing God as a geometric one, especially apt when we consider that Dante does indeed represent the Trinity with a geometric object.

Dante's poem also exhibits a blend of order and chaos, like mathematics does. The poem begins with a dark and confusing scene: Dante writes [2], "In the middle of the journey of our life, I came to myself in a dark wood, for the straight way was lost." Even from the very beginning of the poem, we have all the hallmarks of the "complete confusion and disorder" that is the dictionary definition of chaos. In addition, the designation of "*our* mortal life" seems to imply a sort of universality to this confusion. Yet Virgil's answer to Dante's confusion seems to be a descent into even more chaos, as Virgil says the only feasible way out of Dante's predicament is to journey through Hell itself. Dante agrees to go with him.

At first, it does indeed seem that Hell is nothing but chaos. There is first the sign above the entrance to hell, which says "abandon every hope, you who enter." After Dante enters Hell, it doesn't get much better. We read early on (among other similar descriptions) that Dante comes to a place "where all light is silent, that groans like a sea in a storm when it is lashed by conflicting winds." It seems like Hell is just as chaotic as we would imagine it.

However, Dante seems to intuitively place some sort of order on this chaos. Most significantly, there is the organization of Hell itself. The souls are ordered into 9 circles, with the least sinful in circle 1 and the worst sinners deep in circle 9. The souls are also organized based on the crimes they committed in life: for instance, the lustful are in circle 2 and the violent are in circle 7. Sometimes the circles are divided into smaller subcircles, as circle 7 is divided into 3 further circles based on whom the condemned soul committed violence against. Thus Dante seems to place pockets of order even on what must necessarily be a chaotic place.

Note also the poetic form and rhyme scheme Dante chooses to use. The entire *Divine Comedy* is written in *terza rima*, which translates as a "triple rhyme." The poem is divided into stanzas of three lines. The first and third line rhyme, and the second line rhymes with the first and third line of the following stanza. In addition to the "divine number" of 3 appearing yet again, the stanzas have a beautiful interlocking and orderly structure. Dante invented this form [8]. It serves to show, yet again, how important order in the structure of his poem was to Dante. Despite speaking of the chaotic within his writing, Dante makes sure the framework for it lines up perfectly symmetrically.

Mandelbrot, Fractals, and Chaos Theory

Mandelbrot's view of mathematics is as poetic as Dante's view of the afterlife is mathematical. Mandelbrot claims his motivation for developing his new mathematics was so he could better describe nature. After all, as he writes [4], "clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth." Mandelbrot had to develop a "new geometry of nature" [4] because "nature exhibits not simply a higher degree but an altogether different level of complexity" [4]. This sort of awe at the beauty of nature, as well as the focused effort to better convey its beauty to others, is uncannily similar to the attitude of a romantic poet. Mandelbrot is a poet communicating in mathematical language.

The theory of fractals and dynamical systems, to which Mandelbrot made many contributions, is the ultimate bridge between order and chaos. The quadratic iterator is one of the most striking examples of how the type of math Mandelbrot loved links the two. The iterator's final state diagram is shown below. This diagram is the orbit of the logistic equation

$$f_r(x) = rx(1-x)$$

where $f : [0,1] \to [0,1]$ and r is a positive constant. To obtain and graph the orbits, pick an arbitrary value between 0 and 1 for x, and then for every real number r between 0 and 4, iterate $f_r(x)$ (e.g., calculate $f_r(x)$, plug that value back in for x, and repeat many times.) Then, a thousand or so iterations later, plot the value obtained on the x (vertical) axis to correspond to the chosen r on the horizontal axis. The results are displayed below. (Note r has a domain of 0 to 4, but the image below displays only $2.4 \leq r \leq 4$ so it is easier to read.)

The graph, called a bifurcation diagram, is a fascinating blend of order and chaos. At $2.4 \leq r < 3$, the plot only has one x value per r value. When r is between 3 and 3.44, iterating $f_r(x)$ gives two solutions for x, resulting in an apparent "split" of the graph into two separate branches at $r = 3$.. The number of solutions doubles again to 4 solutions at $r = 3.44$, and the number of solutions continues to double at ever-smaller scales.

After $r = 3.5699456...$, the doubling scenario ends. This r is known as the Feigenbaum point, and after it, the iterator is in a state of full mathematical chaos. The solutions change for every r value. Yet windows of order appear even within this chaos. For instance, note the large white vertical band around 3.82. Here, in the midst of the chaos around it, the equation has only 3 solutions. For r values slightly beyond 3.82, the number of solutions splits to 6, then 12, then 24....and doubles in the same way they did between $r = 2.4$ and $r = 3.5$. Despite the chaos in the picture, the order of this doubling scenario is strongly present. An incredible amount of complexity arises from $f_r(x)$, an equation defined deceptively simply.

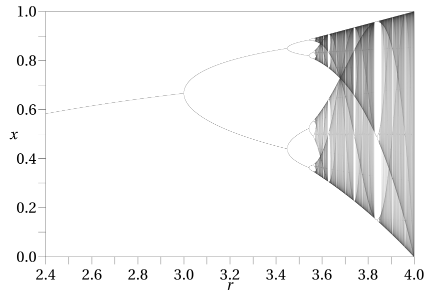

Figure 1 : *The bifurcation diagram. The horizontal axis is r and the vertical axis is our $x \in [0, 1]$.*
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Another example of order within chaos can be seen in the following variations on the classic Sierpinski triangle. (See [5] chapter 5 for more information on the Sierpinski triangle.) These pictures were created with an iterated function system (IFS) which takes the result of an equation and feeds that result back into the original equation.

While strictly deterministic, a very small change in the IFS makes drastically different pictures. These are not chaotic pictures, but their extreme sensitivity to slight differences of rotation angle and translation gives them a sort of ”mock-chaotic” behavior when viewed as a group. These images were re-created from Peitgen et al. [5] using Mathematica.

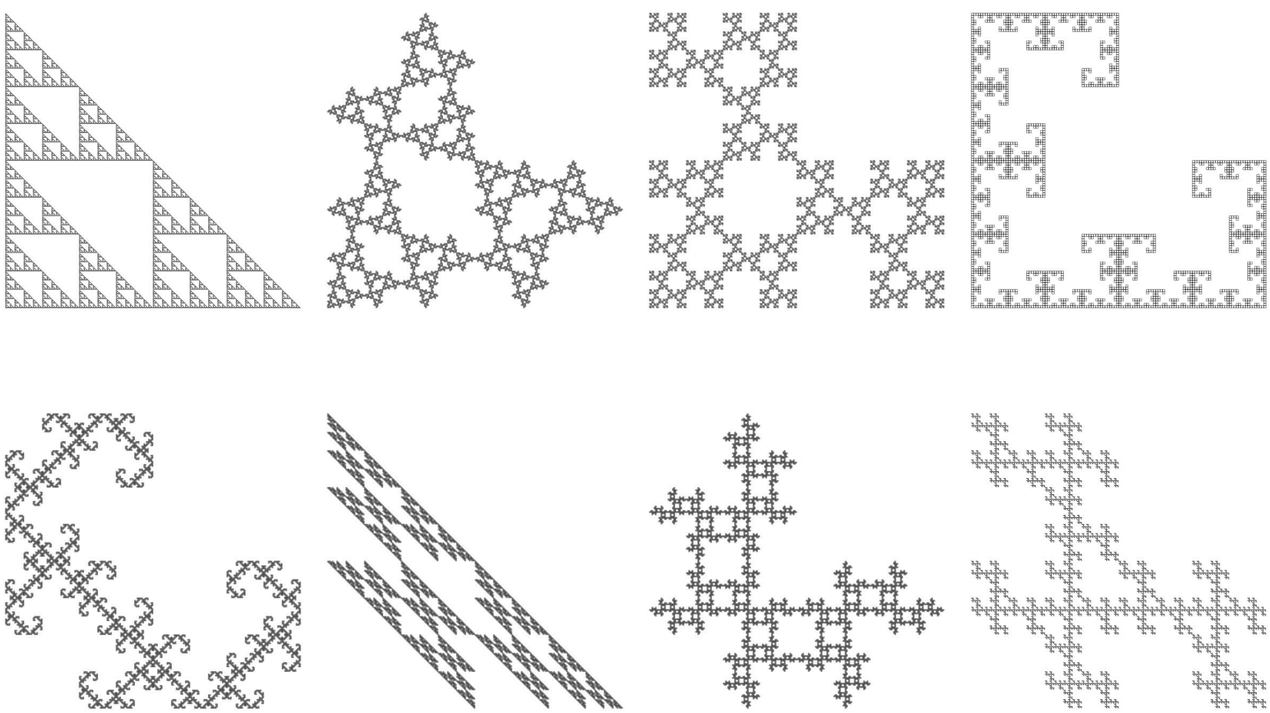

Figure 2 : *8 Sierpinski variations*

The images look extremely different from each other, and while the individual diagrams do not look chaotic, the collection of them together could appear that way. Yet despite their vastly different appearances, the images are intimately connected. They are created by nearly the same equation, differing only by one small rotation angle. In some way, this is like Dante's hell at the beginning of *Inferno*. Even though it looks confusing, unordered, and chaotic, a very present and definite structure and method of ordering rules the scene.

Stoppard's *Arcadia*

Stoppard's play *Arcadia* bridges the work of Mandelbrot and Dante. While Dante explores the realm of mathematics and Mandelbrot acts like a poet, Dante remains primarily a writer and Mandelbrot remains mostly a mathematician. The sole purpose of neither of their works is to bridge the gap between math and literature, and the order and chaos connections which arise from their work do so from the work itself, not because their main focus is to examine the order and chaos connection. While the order and chaos appears with both of them, it is a natural by-product, rather than the goal, of their work. Stoppard differs from them on both of these counts. He deliberately seeks to combine both the math and literature, order and chaos dichotomy.

Consider the bridge between math and literature. This is visible even in the associations of the play's characters: Valentine, a mathematician, and Hannah, a historian, work closely together. Thomasina, a bright young math student, tries to catch the attention of Lord Byron, the famous poet. In addition, letters and the written word are essential to get any type of mathematics done. Valentine can only work out Thomasina's equations after reading what she had previously written about them. Lack of writing was what stopped Thomasina from finishing her project: Valentine notes that she "ran out of pages." Math and literature feed each other. Math can't be done without literature, and Hannah can't write her paper about history without understanding Thomasina's mathematics.

Stoppard talks directly about both the technicalities of fractals and the motivation behind their development. Thomasina asks her tutor Septimus, "If there is an equation for a curve like a bell, there must be an equation for one like a bluebell, and if a bluebell, why not a rose?...why do your equations describe only the shapes of manufacture?" Her questions directly echo Mandelbrot's observations [4] that "clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth." Mandelbrot's quest to develop a "new geometry of nature" strongly echoes Thomasina's attempt to fabricate a "New Geometry of Irregular Forms." Either Stoppard has read Mandelbrot's book, or Stoppard's line of questioning accidentally parallels Mandelbrot's. Either way, Stoppard makes a powerful statement about the connection between order and chaos in nature: nature is ordered and beautiful, and yet it is indescribable, somehow unknowable, with the tools of classic geometry. As Thomasina and Mandelbrot both realize, more is needed.

In addition to the fractal mathematics, order and chaos are a huge part of the characters' lives as well. Septimus notes that "we must stir our way onward, mixing as we go, disorder out of disorder out of disorder..." The play's brilliant ending twist is the ultimate combination of order and chaos, the perfectly explained and the perfectly unexplainable. A death that is tragic and incomprehensible to Septimus is the final piece in an orderly historical puzzle for Hannah. Order versus chaos is only separated by perspective and scale, just like it is in the mathematics.

Dante describes the interplay of order and chaos in a metaphysical context. Mandelbrot describes the interplay in a mathematical context. Stoppard does both. When Valentine points at his computer and comments "see? In an ocean of ashes, islands of order" he is speaking both of the mathematics and of life itself.

Fractal examples inspired by *Arcadia*

In *Arcadia*, Valentine shows Hannah his computer pictures, saying "It's Thomasina's. I just pushed her equations through the computer a few million more times further than she managed to do with her pencil." While Stoppard does not give us Thomasina's equations so it is not possible to exactly replicate Valentine's work, it is possible to use Mathematica and some educated guesses to approximate the math Stoppard describes in various parts of the text.

Ferns

Valentine describes a mathematical process to Hannah, saying "If you knew the algorithm and fed it back say ten thousand times, each time there'd be a dot somewhere on the screen. You'd never know where to expect the next dot. But gradually you'd start to see this shape, because every dot will be inside the shape of a leaf" Valentine is describing the Chaos Game, a way of randomly creating a fractal object by placing a dot on the screen whose location is assigned based on a set of probabilities. There are two ferns pictured below. The first was created deterministically, through a shrink, rotate, and translate method similar to the Sierpinski variations above. The pictured points (using small squares instead of points for better visibility) are the only ones that would plot before the program began to run slowly. (The 7-iteration image shown took about 5 minutes of runtime to create; an 8-iteration fern had no results after about half an hour.) There are about 15,000 points plotted. The second was created using the chaos game, and plotted 100,000 points in under 2 seconds. It is known as the Barnsley Fern after the mathematician Michael Barnsley, who invented the chaos game. The random version not only still gives a picture of a fern, but gives a *better* picture of the fern, and in a faster runtime, than the deterministic version.

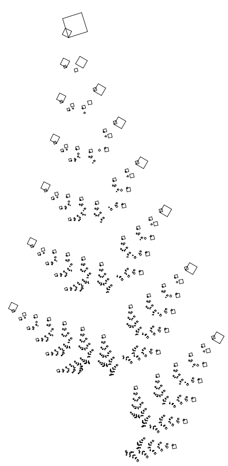

Figure 3: *Creating a fern using the IFS method.*

Figure 4: *Creating a fern using the chaos game*

Leaves

One of the questions Thomasina poses throughout *Arcadia* is why there is no geometry to describe nature. She questions Septimus, saying "Why do your equations only describe shapes of manufacture?" In an attempt to fix what she perceives as this error, she picks up an apple leaf and declares "I will plot this leaf and deduce its equation."

In an attempt to complete or at least contribute to what Thomasina couldn't do, I tried to plot some natural objects using Mathematica. This is a sort of closing of the circle in the relationship between mathematics and English. Stoppard's play was inspired by Mandelbrot's mathematics, and then a mathematician was inspired to make math based on the play. A very simple 4-equation IFS gives the rough outline of a tree (figure 5). All rotation angles are 20 degrees and all scaling ratios are a little more that .5.

Different scaling ratios and rotation angles make different objects. Changing nothing but making the scaling ratio different for the right-facing top branch gives the "wind-blown," more sparse tree of figure 6.

Keeping the scale the same but doubling the rotation angle and adding a fifth equation (moves halfway up the original line and rotates right) gives the denser symmetrical tree of figure 7.

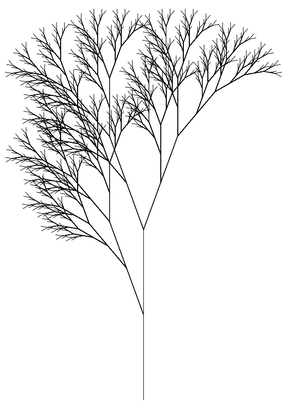

Figure 5 : *An IFS tree.*

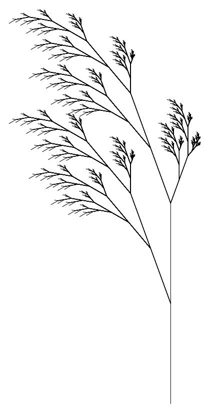

Figure 6 : *Another IFS tree.*

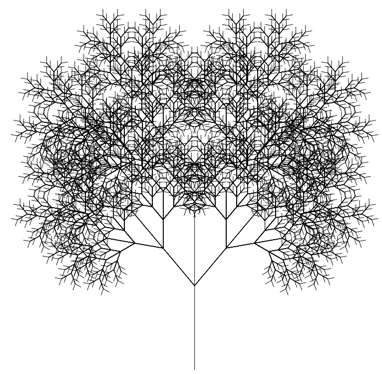

Figure 7 : *An "dense" IFS tree.*

Thomasina was fascinated with leaves as well. Figure 8 is an attempt at a maple leaf.

Finally, one more symmetrical, sparse-like leaf is shown in figure 9. Figure 10 is a fractal with the same IFS, except the starting object is a square instead of a line, which gives an interesting shading effect at these relatively low iterations.

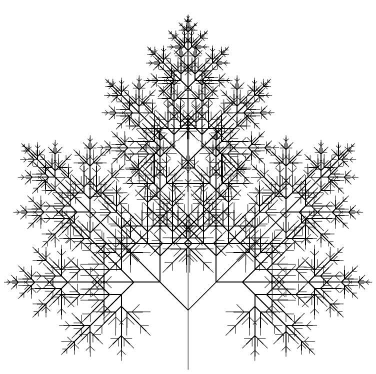

Figure 8 : *A maple leaf IFS*

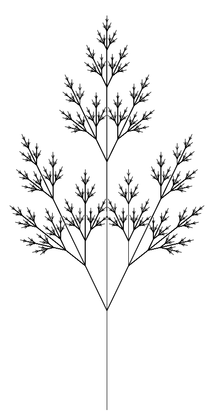

Figure 9 : *Another IFS tree.*

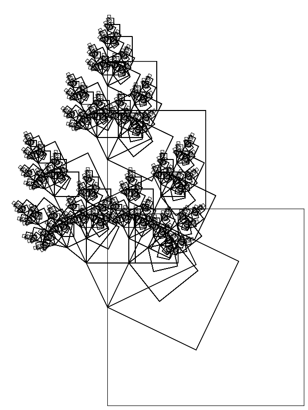

Figure 10 : *The tree of figure 9 with a different starting object.*

What Thomasina tried and failed to do (and what Valentine says "you'd have to be insane" to try) is now a matter of plugging three equations into Mathematica.

Conclusion

Order and chaos, math and literature are concepts often seen as total and complete opposites. Yet really, they are anything but. The concepts are inseparable. Great literature can be inspired from great math, and great math can be inspired from great literature. Viewing them as complete opposites, either in academia or in life, is counterproductive to the advancement of both fields. Similarly, seeing order and chaos as completely opposites concepts is both uninformative and inaccurate, as the two intertwine in interesting and informative ways both in math and in art.

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