Fractional Beauty

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Abstract

Superb patterns can be realized by playing with continued fractions within the constraints of plane symmetries. Two approaches are presented: one introducing *deflection lines*, the other one introducing *direction* in plane symmetries.

In this paper I present two approaches to generate patterns within plane symmetries, both based on the introduction of a ratio. In the first approach a ratio is introduced by entering *deflection lines*, resulting in stripe patterns, which are built up of layered spirals. In the second approach a ratio is introduced by entering a *direction*, resulting in crystalline patterns. Converting these approaches to computer programs was rather easy. Once I had done that, thousands of structures could be generated in no time. However, I could not get a grip on the relation between introduced ratio and emerged structure, until I discovered that the ratio had to be rewritten as a continued fraction. The reason why this 'trick' was working remained for a long time a mystery. At a certain moment however I discovered that the evoked complexity could be seen as a series of events that perfectly matched the structure of the continued fraction. These insights have grown in practice and formal mathematical proofs are not presented here.

Introducing a ratio by means of deflection lines

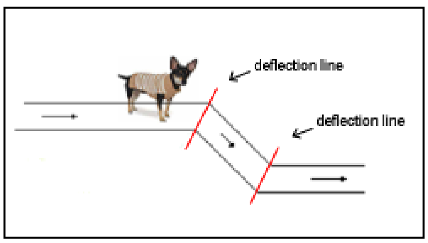


Figure 1 : *Deflection lines*

We consider a stripe as a path running in a certain direction. Deflection lines mark the places where the path changes its course (Figure 1). Hitting a deflection line, the entering and the exiting path have the same angle with it, so the path maintains constant width. The part of a deflection line which is traversed by the path is the unit of length. The real length of this unit increases as the angle of bending becomes sharper. Plane symmetries in which only rotations occur provide a suitable field to generate stripe paths by means of deflection lines. In particular those containing 6-fold rotations (p6), those containing 4-fold rotations (p4) and those containing only 3-fold axes (p3). Figure 2 shows a stripe path determined by two families of deflection lines with mutually variable length within p6. The length ratio is 5/17 in this case. A grid of triangles underlies the design process. The borders of the stripe path coincide with lines of the grid. In one of their ends, deflection lines with the same length meet each other in rotation axes: lines with a length of 17 units in 3-fold axes; lines with a length of 5 units in 6-fold axes. At their other end, deflection lines of different length meet each other at an angle of 90º. This system of deflection lines divides the plane into repeating units. Each unit is divided in stripes. Stripes in different repeat units unite and form stripe paths which have the nature of closed circuits. The path, in crossing S(shorter) and L(longer) deflection lines, generates the following series of S-and L-events, starting at the place and in the direction indicated by the arrow: SLLLSLLLSLLLLSLLLSLLLL. The series repeats itself half-way along the path. The structure of the series can be presented in a layered diagram (Figure 4a). At each layer there is an alternation of the elements *n* and *1*. In the first layer, these elements indicate the number of L- events between two S-events, being always *n* or *n+1*, whereby *1* is recorded as a separete element besides *n*. After the first layer, the alternation of *n* and *1* continues, but now these elements indicate the number of *n*-elements between two *1*-elements at the preceding level. Over the different layers, these patterns of alternation of *n* and *1* realize the best solution for an even spreading of S- events on L-events. The value *n* for the different layers can be derived from the S/L-ratio by applying a proces of repeated division to it. In the first division you divide L by S. In the following

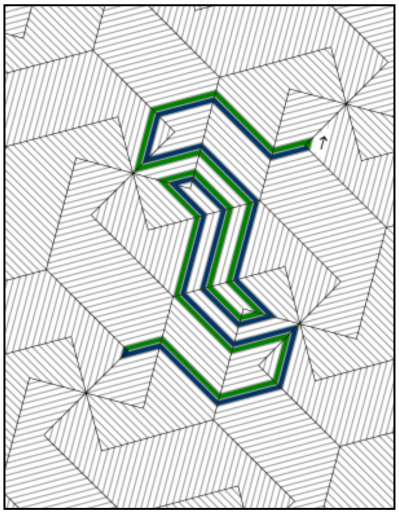


Figure 2: *Stripe path 5/17.*

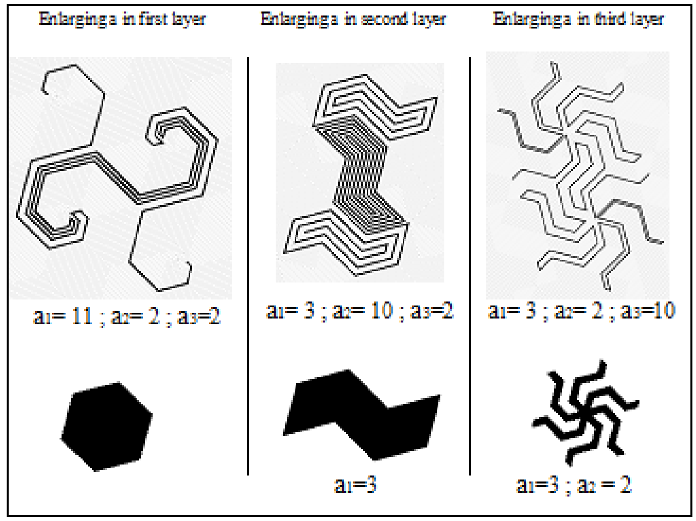
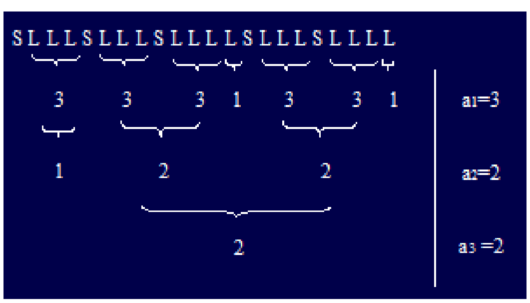


Figure 3: *Three spiral layers in stripe path 5/17.*

divisions you divide the divisor of the preceding division (becoming the dividend now) by its remainder (becoming the divisor now), until the remainder is 0. The result is a series of partial quotients which are indicated as a_1, a_2, a_3, .., which can be presented in the format of a continued fraction (Figure 4b). Figure 4a shows that the structure diagram of the ratio 5/17 perfectly matches with the series of *a*-values in the continued fraction of 5/17: the number of layers in the structure diagram corresponds with the number of *a*-values in the continued fraction; the successive values for *n* in the structure diagram corresponds with the successive values for *a* in the continued fraction.



a

b

general model:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

application to 5/17:

$$\cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}}$$

Figure 4: *Relation between the generated S/L-series and the continued fraction.*

Layered spirals. A stripe path, like that generated by 5/17, in essence is a layered spiral, the series of *a*-values being indicative of the spiral characteristics. The number of *a*-values in the series represents the number of layers in the spiral. The spiral winds itself in the first and third layer around 6-fold axes of rotation and in the second layer around 2-fold axes of rotation. That becomes clear when you increases the value *a* of each of these layers (Figure 3).The higher the value *a* of a layer, the more the spiral winding continues at that layer. In the first layer spiral winding is around the perimeter of a hexagon, in the second around the perimeter of a 'zigzag' and in the third around perimeter of a spider-like figure. The perimeter around which spiral winding takes place in a certain layer, is delivered by the stripe path represented by the *a*-series that ends just before the *a* value of that layer. So the spider-like figure, which

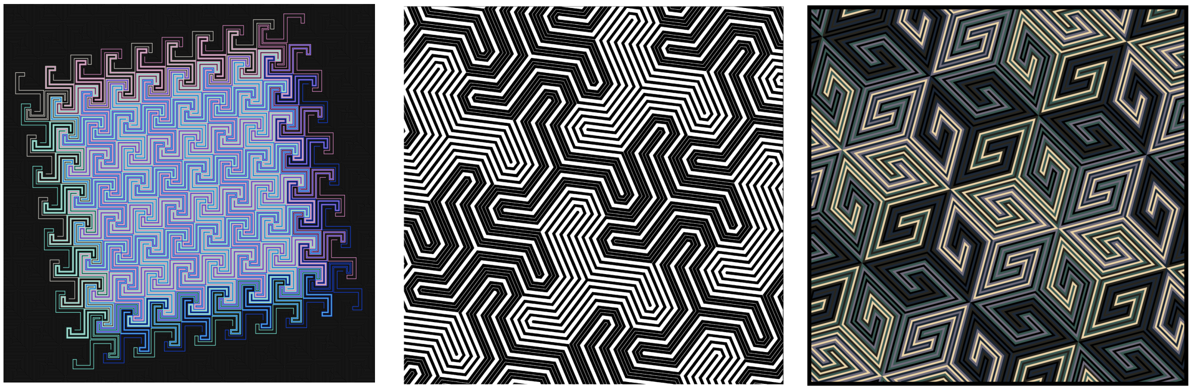


Figure 5: *Layered spiral structures in different types of plane symmetry.*

delivers the perimeter around which spiral winding takes place in the third layer, is represented by the series: a_1=3 ; a_2=2. As to spiral winding in the first layer, there is no preceding series of *a*-values. Here the hexagon, representing the pre-spiral stage, delivers the perimeter for spiral winding. Besides the number of layers and the degree and the course of spiral winding per layer, also the potentialities for beautiful color mixing can be derived from the continued fraction (see attachment). So, by means of continued fraction intriguing patterns can be designed in a foreseen way (Figure 5).

Introducing a ratio by means of direction.

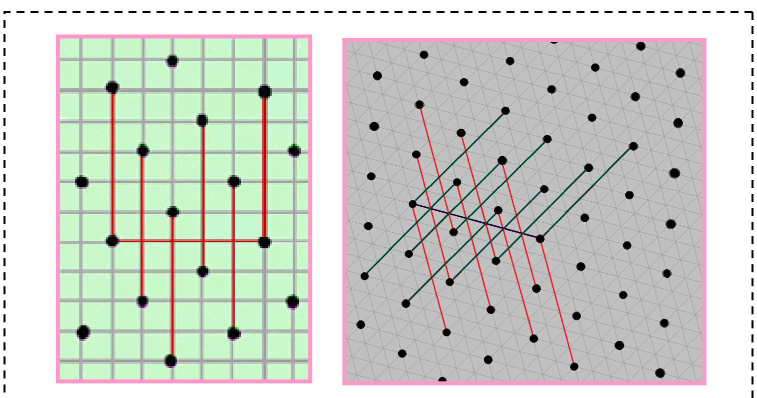


Figure 6: *Introducing direction in p4 and p6.*

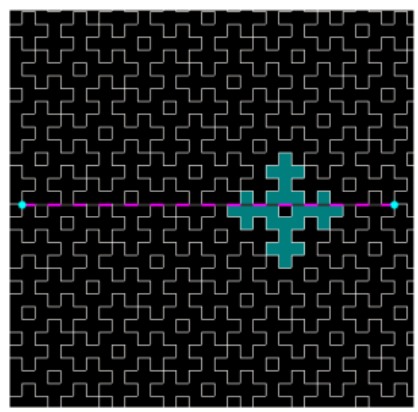


Figure 7: *Emergence of crystals.*

In this approach we introduce a direction within plane symmetries of the type of P4 and P6. In both you can draw a line between two rotation axes in a certain direction. The direction can be expressed in a ratio relative to the repeat units, which have rotation axes as their corner points. The direction results in a pattern of mutually intersecting lines running in two or three orientations of that direction, dependent on the type of symmetry (Figure 6 illustrates the ratio 1/2). This intersection pattern is the base for pattern generation. Every line is divided in equal sections by lines running in an other orientation. By alternately omitting and including a section on the lines that intersect, the complexity of the intersection pattern is directly transformed to crystal figures (Figure 7 illustrates 2/5). Also in this approach, the relation between introduced ratio and evoked structure can only be understood when you rewrite the ratio as a continued fraction. Every direction, like 2/5 in Figure 8a, in its traversing horizontal and vertical sides of repeat units (Figure 8a), evokes a series of S- and L-events which matches perfectly the series of *a*-values in the continued fraction of S/L. The intersection pattern of 2/5 roughly has the same S/L-structure (Figure 8b).

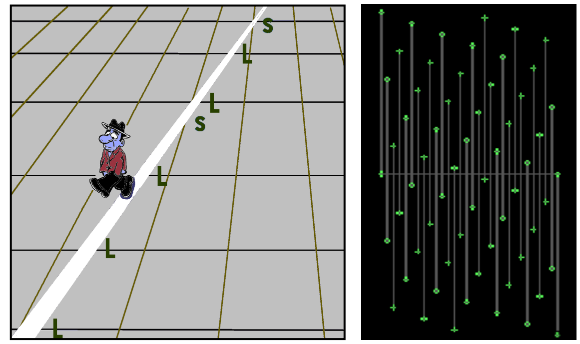


Figure 8: *S/L- series of 2/5.*

Crystal figures. Every ratio results in a crystal figure, the characteristics of which can be derived from the *a*-values in the continued fraction. On the base level, every value in the series of *a*-values is 2 and yields an extra level in the 'crossing-up' of the perimeter of the square (Figure 9a). If a value *a* is made greater than 2, provided that it remains even, repetition occurs in the perimeter of the figure. We can illustrate this for the crystal figure represented by 2 2 2 (Figure 9a). If we increase the first value 2, all three layers in 'crossing up' are involved in repetition (Figure 9b). If we increase the second value 2, only two layers are involved in repetition. And if we increase the third, only one level is involved. As soon as a value in the series of *a*-values is odd, the process of 'crossing up' stops and ring formation starts (Figure 9c). The size of that odd value determines the number of columns in ring formation. The next value in the series of *a*-values determines the number of rings per center. After these

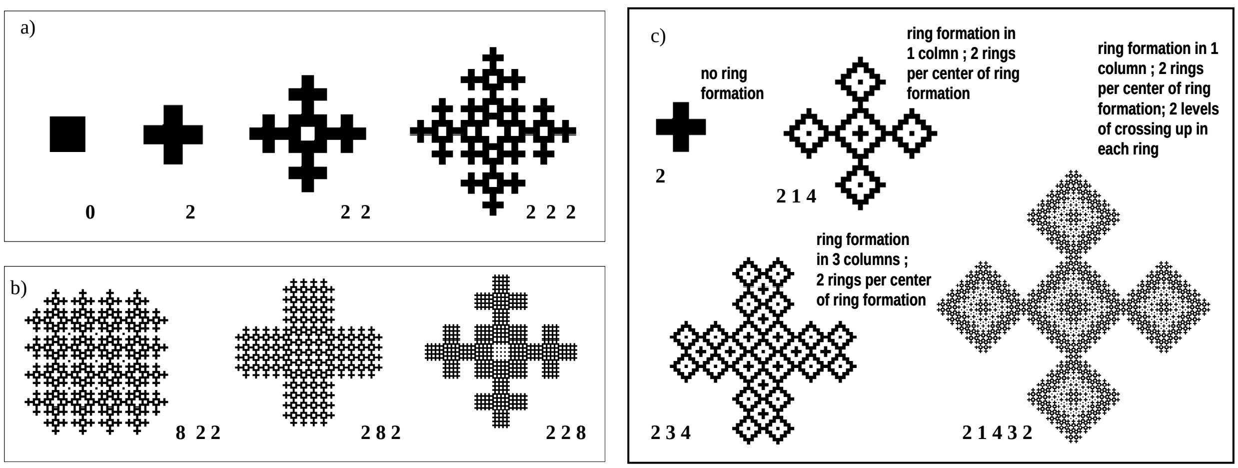
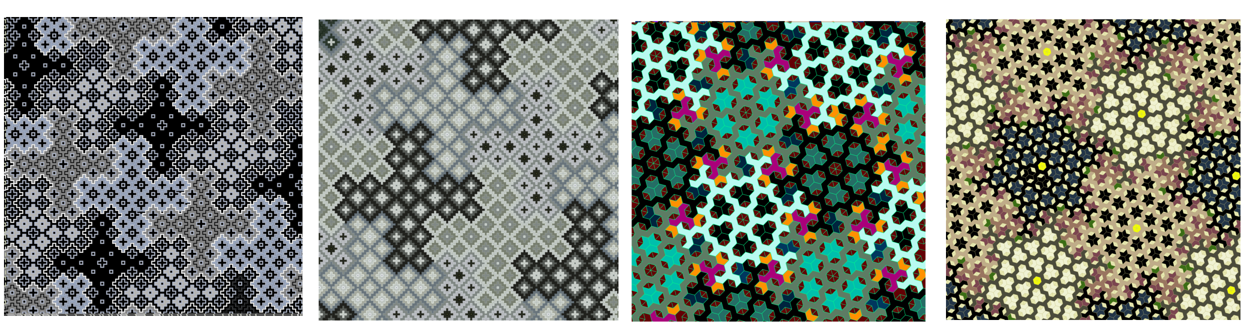


Figure 9: *Relation between the a-values in the continued fraction and the pattern characteristics*.

two *a*-values, that determine the characteristics of ring formation, subsequent values in the *a*-series result in continuation of the process of 'crossing up', but now at the level of the rings. Three *a*-values further on in the series, the newly appearance of an odd value again causes the start of ring formation. Beautiful patterns can be generated in this way (Figure 10a), but the results are also impressive if there are 3 orientations in the direction (Figure 10b).



a

b

Figure 10: *Patterns generated by introducing direction in a plane symmetry*